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Exercise sheet 6

Partial Differential Equations

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Exercise 19: Examples of quasilinear PDEs of first order

Solve by the method of characteristics the following quasilinear PDEs. Describe, if possible, the characteristic vector fields and Cauchy data geometrically. What is the maximal domain of definition of the solution? How can one recognize this already from the characteristics?

a) $x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u, \quad u(x_1, x_2, 0) = f(x_1, x_2)$ with $f \in C^1(\mathbb{R}^2)$.

b) $x_2 u_{x_1} - x_1 u_{x_2} = 4x_1 x_2, \quad u(x_1, 0) = a x_1^2$ with $a \in \mathbb{R}$.

c) $u_{x_1} + u_{x_2} = u^2, \quad u(x_1, 0) = g(x_1)$ with $g \in C^1(\mathbb{R})$.

d) $\sum_{k=1}^n x_k u_{x_k} = \alpha u \ \ (\alpha \neq 0), \quad u(x_1, \dots, x_{n-1}, 1) = h(x_1, \dots, x_{n-1})$ with $h \in C^1(\mathbb{R}^{n-1})$. What is the relation between $u(\lambda x)$ and $u(x)$ for arbitrary $\lambda > 0$?

Exercise 20: A linear PDE solvable in closed form

Show by geometric considerations that the general solution of $x u_x - y u_y = 0$ for $x, y > 0$ is $u(x, y) = f(xy)$. Find the solution whose graph contains the line $u = x = y$. What happens to the initial value problem if one prescribes u on the curve $x = y^{-1}$?

Exercise 21: Picone's example

Let u be a solution of

$$a(x, y) u_x + b(x, y) u_y = -u$$

of class C^1 in the closed unit disk Ω in the xy-plane. Let $a(x, y)x + b(x, y)y > 0$ on the boundary of Ω. Prove that u vanishes identically.

Hint: Show that $\max_\Omega u \leq 0$, $\min_\Omega u \geq 0$, using conditions for a maximum at a boundary point.