PROF. DR. BERND SCHMIDT
DR. JOHANNES GIANNOULIS

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EXERCISE SHEET 5

Partial Differential Equations

Winter semester 2009/10

Exercise 16: Time-asymptotics of the three-dimensional wave equation

Let u be the solution of the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u = g,\ u_t = h & \text{on } \mathbb{R}^3 \times \{0\} \end{cases}$$

for smooth g, h with compact support. Show that there exists a $C \geq 0$ such that

$$|u(x,t)| \leq C/t, \quad \forall x \in \mathbb{R}^3, t > 0.$$

Exercise 17: Characteristic forms of differential operators

Determine and describe the characteristic forms and varieties of the following differential operators L. Are the corresponding equations $Lu = f$ of elliptic/hyperbolic/parabolic type?

a) $L = \partial_t^2 - \Delta$ on $\mathbb{R}^{n,1}$. How can you formulate your result by use of Exercise 13?

b) $L = \partial_1^2 + 2\partial_1\partial_2 - 2\partial_1\partial_3 + 2\partial_2^2 + 6\partial_3^2$ on $\mathbb{R}^3$.

c) $L = \partial_1\partial_2 - \partial_1\partial_3 + \partial_1 + \partial_2 - \partial_3$ on $\mathbb{R}^3$.

Exercise 18: The 1-d Fourier-method for periodic boundary-conditions (part II)

This exercise is a continuation of Exercise 15. Our aim here is to clarify under which conditions one can solve the boundary-value problem given there by the Fourier-method.

a) Show first the following (already known) assertions:
Let $L^2(S^1) := \{f : \mathbb{R} \to \mathbb{C} \text{ measurable} : f \in L^2[0,2\pi] \text{ and } f(x+2\pi) = f(x) \text{ for almost every (a.e.) } x \in \mathbb{R}\}$ with the scalar product $\langle \cdot\,, \cdot \rangle_{L^2[0,2\pi]}$ be the Hilbert space of 2π-periodic L^2-functions. Obviously $L^2(S^1) \cong L^2[0,2\pi]$.
Let $f \in L^2(S^1)$ be continuous and almost everywhere differentiable with derivative $f' \in L^2(S^1)$. Then the following is true:

(i) The Fourier-coefficients $\widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-ikx}\,dx$ and $\widehat{f'}(k)$ are related by

$$\widehat{f'}(k) = ik \cdot \widehat{f}(k), \ \forall k \in \mathbb{Z}.$$

(ii) The Fourier series of f converges normally, i.e. $\sum_{k\in\mathbb{Z}} |\widehat{f}(k)| < \infty$, and in particular uniformly to f.

b) Consider now again the periodic boundary-value problem (3) of Exercise 15 for given 2π-periodic functions $g \in C^2(S^1, \mathbb{C})$, $h \in C^1(S^1, \mathbb{C})$ (defined as above) and the ansatz for the solution developed there:

$$u(x,t) := \widehat{g}(0) + t \cdot \widehat{h}(0) + \sum_{k \neq 0} \left(\widehat{g}(k) \cos(kt) + \frac{\widehat{h}(k)}{k} \sin(kt) \right) e^{ikx}.$$

Show by use of a) the following theorem:

If the Fourier series of g'' and h' converge normally, then $u \in C^2([0, 2\pi] \times [0, \infty))$ and u is a solution of the periodic boundary-value problem (3) of Exercise 15.

The assumption of the theorem is satisfied, if g'' and h' are continuous and almost everywhere differentiable with derivatives $g''', h'' \in L^2(S^1)$.