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Exercise sheet 4

Partial Differential Equations

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Exercise 13: Lorentz-invariance of the wave equation

The set $\mathbb{R}^{n,1} := \{(x,t) \mid x \in \mathbb{R}^n, t \in \mathbb{R}\} \cong \mathbb{R}^{n+1}$ equipped with the (degenerate) *Minkowski inner (or scalar) product*

$$\langle \cdot\,, \cdot \rangle : \mathbb{R}^{n,1} \times \mathbb{R}^{n,1} \to \mathbb{R},\ \langle (x,t),\, (y,s) \rangle := -x^T y + ts$$

is called the (n+1)-dimensional *Minkowski space*.

a) The linear isometries with respect to (w.r.t.) the Minkowski inner product, i.e., the linear maps $L : \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ with

$$\langle L(x,t),\, L(y,s) \rangle = \langle (x,t),\, (y,s) \rangle \quad \forall (x,t),(y,s) \in \mathbb{R}^{n,1},$$

are called *Lorentz transformations*. Show that the set $O(n,1)$ of all Lorentz transformations is a subgroup of $GL(n+1,\mathbb{R})$. Give a non-trivial example of a Lorentz transformation.

b) The trace $\mathrm{tr}(A)$ of a linear map $A : \mathbb{R}^{n,1} \to \mathbb{R}^{n,1}$ is defined by

$$\mathrm{tr}(A) := -a_{11} - \ldots - a_{nn} + a_{n+1,n+1},$$

where (a_{ij}) is the representation matrix of the map A w.r.t. the usual standard basis $\{e_1, \cdots, e_{n+1}\}$ of $\mathbb{R}^{n,1}$. Show that the trace is invariant under Lorentz transformations:

$$\mathrm{tr}(L^T A L) = \mathrm{tr}(A) \quad \forall L \in O(n,1).$$

Hint: Show first that $\mathrm{tr}(A) = \mathrm{tr}_E(gA)$, where g is the representation matrix of the Minkowski inner product in $\mathbb{R}^{n+1}$ and tr_E is the Euclidean standard trace.

c) Show that for all $u \in C^2(\mathbb{R}^{n,1})$ and all $L \in O(n,1)$:

$$(\partial_{tt}^2 - \Delta)(u \circ L) = (\partial_{tt}^2 u - \Delta u) \circ L.$$

What does this mean in particular for solutions of the wave equation $u_{tt} - \Delta u = 0$?

Exercise 14: Equipartition of energy

Let $u \in C^2(\mathbb{R} \times [0,\infty))$ be a solution of the one-dimensional boundary-value problem

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty), \\ u = g,\ u_t = h & \text{auf } \mathbb{R} \times \{0\}, \end{cases}$$

where g,h have compact support. $k(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)\,dx$ and $p(t) := \frac{1}{2}\int_{-\infty}^{\infty} u_x^2(x,t)\,dx$ are called, respectively, the *kinetic* and the *potential* energy of the solution u.

Show:

a) $k(t) + p(t)$ is constant in t,

b) $k(t) = p(t)$ for sufficiently large times t.

Exercise 15: The Fourier method for solving partial differential equations

Let $(V, \langle \cdot\,, \cdot \rangle)$ be a (complex) pre-Hilbert space and $A : V \to V$ a linear map. At first, let $\dim V = n < \infty$ and let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of V, consisting of eigenvectors of A for the eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$.

a) Consider the following linear ordinary differential equation (ODE) on V:

$$\frac{d}{dt} w = Aw, \quad w(0) = g \in V. \tag{1}$$

Verify the following result (which should be already known to you):
Setting $a_i := \langle g, v_i \rangle$, one has $g = \sum_{i=1}^n a_i v_i$ and

$$w(t) = \sum_{i=1}^n a_i e^{\lambda_i \cdot t} v_i$$

is the uniquely determined solution of (1).

b) What is the analogous statement for the initial-value problem

$$\frac{d^2}{dt^2} w = Aw, \quad w(0) = g, \; \frac{d}{dt} w(0) = h \in V? \tag{2}$$

Why does the additional term "$\frac{d}{dt} w(0) = h$" appear here?

We are looking now for solutions of the wave equation with periodic boundary conditions

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{in } [0, 2\pi] \times (0, \infty), \\ u = g, u_t = h & \text{on } [0, 2\pi] \times \{0\}, \\ D_x^\alpha u(t, 0) = D_x^\alpha u(t, 2\pi) & \text{for } t \in [0, \infty), |\alpha| \le 2. \end{cases} \tag{3}$$

To this end, let $V := \{v \in C^2([0, 2\pi], \mathbb{C}) \mid D^\alpha v(0) = D^\alpha v(2\pi) \text{ für } |\alpha| \le 2\}$ and $g, h \in V$.

c) Now we view the wave equation (3) as a linear ODE (2) in the (complex) pre-Hilbert space $(V, \frac{1}{2\pi} \langle \cdot\,, \cdot \rangle_{L^2[0,2\pi]})$ with the operator

$$A = \frac{d^2}{dx^2} : V \subset L^2[0, 2\pi] \to L^2[0, 2\pi].$$

Which initial conditions should we require? What about the boundary conditions in (3)?

d) Show that $\{e^{ikx}\}_{k \in \mathbb{Z}}$ is an orthonormal basis of V consisting of eigenvectors of A, i.e., in particular, that each $v \in V$ can be represented in the form $v(x) = \sum_{k \in \mathbb{Z}} c_k \, e^{ikx}$ (where the convergence of the series is to be understood in the corresponding norm). Determine the corresponding eigenvalues.
Hint: Recall what you know about Fourier series.

e) Consider now the wave equation (3) for the initial data $g(x) = x^2(x - 2\pi)^2$ and $h \equiv 0$. Verify that g, h are indeed elements of V. Use now the ansatz from b) – at first formally[1] – with the basis from d) and the initial values g and h. Show that you obtain in that way indeed a solution! Plot the solution!

[1] "formally" because here you should obtain an (infinite) series (and not a – finite – sum) of which at first you do not know whether it converges or not.