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SHEET 3

Partial Differential Equations

Winter semester 2009/10

Exercise 9: A convergence theorem for harmonic functions

Let (u_k) be a monotonically increasing sequence of harmonic functions in a connected domain $U \subset \mathbb{R}^n$ and suppose that for some point $y \in U$ the sequence $(u_k(y))$ is bounded. Then the sequence converges uniformly on any bounded subdomain $V \subset\subset U$ to a harmonic function.

Exercise 10: Green's function for a half-space

Determine Green's function for the half-space $\mathbb{R}^n_+ := \{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$:

$$G(x,y) = \Phi(y-x) - \varphi^x(y), \quad x, y \in \mathbb{R}^n_+,\ x \neq y.$$

Hint: Use reflection at the hyperplane $x_n = 0$ in order to obtain $\varphi^x(y)$.

Exercise 11: The heat equation in $\mathbb{R}^n$

In class it was shown that

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\,dy + \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)f(y,s)\,dy\,ds$$

(where $\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{\|x\|_2^2}{4t}}$ is the fundamental solution) is a solution of the Cauchy-Problem

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{0\}, \end{cases} \tag{1}$$

provided suitable conditions on f and g are fullfiled. In the following, we consider these conditions as satisfied and discuss the given solution in more detail.

a) Let $f, g \in L^1(\mathbb{R}^n)$. Show that the *total amount of heat* $Q(t)$ satisfies

$$Q(t) := \int_{\mathbb{R}^n} u(x,t)\,dx = \int_{\mathbb{R}^n} g(y)\,dy + \int_0^t \int_{\mathbb{R}^n} f(y,s)\,dy\,ds.$$

Give an interpretation of your result, in particular for the case $f \equiv 0$.

b) Assume in the following $f \equiv 0$. Show that

$$u(x,t) \to 0 \ \text{ uniformly with respect to } x \text{ as } t \to \infty.$$

c) Determine an explicit formula for the solution $u(x,t)$ of (1) for the initial distribution

$$g(x) = e^{-\|x\|_2^2}, \ x \in \mathbb{R}^n.$$

Sketch the solution at various times t.

Exercise 12: The heat equation in a bounded region

Let $U \subset \mathbb{R}^n$ open and bounded with C^1-boundary. Consider the boundary-value problem

$$\begin{cases} u_t - \Delta u = f & \text{in } U \times (0,\infty), \\ \partial_\nu u = g & \text{on } \partial U \times (0,\infty), \\ u = h & \text{on } U \times \{0\}, \end{cases} \tag{2}$$

for given continuous functions f, g, h.

a) What is the physical interpretation of the boundary condition "$\partial_\nu u = g$ on $\partial U \times (0,\infty)$"?

b) Show that the problem (2) has at most one solution $u \in C^2_1(\overline{U} \times [0,\infty))$.
Hint: Consider for the difference $w = u - \tilde{u}$ between two solutions the *energy functional* $E(t) = \int_U w^2(x,t)\,dx$. Show that $E \equiv 0$.

Let now $u \in C^2_1(\overline{U} \times [0,\infty))$ be a solution of (2).

c) Prove that the total amount of heat $Q(t) = \int_U u(x,t)\,dx$ is given by

$$Q(t) = \int_U h(x)\,dx + \int_0^t \left(\int_U f(x,s)\,dx + \int_{\partial U} g(\xi,s)\,dS(\xi) \right) ds.$$

Give a physical interpretation of your result, in particular for the case $f, g \equiv 0$.

d) Let $f, g \equiv 0$. Consider the asymptotic behavior of the energy functional $E(t) = \int_U u^2(x,t)\,dx$ and its derivatives up to second order as $t \to \infty$. Derive from your results that

$$\|u\|_{L^2(U)} \le \|h\|_{L^2(U)} \;\; \forall t \ge 0 \;\text{ and }\; \|u_{x_i}\|_{L^2(U)} \to 0,\; t \to \infty.$$

Remark: The last result implies that u approaches for $t \to \infty$ a function that is constant on the connected components of U.