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Sheet 2

Partial Differential Equations I

Winter semester 2009/10

Exercise 5: Banach space of harmonic functions

Let $U \subset \mathbb{R}^n$ be open and bounded. Show that $(\{u \in C(\overline{U}) \mid u \text{ harmonic in } U\}, \|\cdot\|_\infty)$ (with $\|u\|_\infty := \sup_{x\in\overline{U}} |u(x)|$) is a Banach space.

Hint: You may use the fact that $(C(\overline{U}), \|\cdot\|_\infty)$ is a Banach space.

Exercise 6: Neumann boundary-value problem for the Poisson equation

Let $U \subset \mathbb{R}^n$ be open and bounded with a C^1-boundary and outer unit normal ν, as well as $f \in C(\overline{U})$, $g \in C(\partial U)$. Show that if the Poisson equation with *Neumann boundary conditions*,

$$\begin{cases} -\Delta u = f & \text{in } U \\ \partial_\nu u = g & \text{on } \partial U \end{cases},$$

has a solution $u \in C^2(\overline{U})$, then

$$\int_U f\,dx + \int_{\partial U} g\,dS = 0.$$

Exercise 7: Harnack's inequality

Let $U \subset \mathbb{R}^n$ be open. Show that for every connected $V \subset\subset U$ there exists a $C > 0$ such that

$$\sup_V u \leq C \inf_V u$$

for all functions $u \geq 0$, which are harmonic in U.

Exercise 8: Faraday cage

The Poisson equation of electrostatics

$$\rho = -\Delta\varphi \tag{1}$$

describes the relation between the *electric charge density* ρ and the *electrostatic potential* φ. Suppose $U \subset \mathbb{R}^3$, U open with C^1-boundary, is a bounded region, surrounded by a conducting material ∂U. Let $\rho \in C_c(\mathbb{R}^3)$ be a compactly supported charge density, which vanishes on $\overline{U}$, i.e. $\operatorname{supp}(\rho) \subset \mathbb{R}^3 \backslash \overline{U}$, and let $\varphi \in C^2(\mathbb{R}^3)$ be a corresponding potential, which satisfies (1). According to the theory of electrostatics, the potential φ has to be constant on ∂U:

$$\varphi|_{\partial U} = \text{ const.}$$

Show that this implies that the electric field vanishes in U:

$$\nabla\varphi|_U = 0.$$