EXERCISES PDE 31.10.12-02.11.12

1. Exercise

Let $U \in \mathbb{R}^N$ be a bounded open set. We say that $v \in C^2(\bar{U})$ is **subharmonic** iff $-\Delta v \leq 0$ in U.

(a) Prove that subharmonic functions enjoy the following form of the mean-value property: for every $x \in U$, for every $r > 0$ such that $B(x,r) \subset U$

$$v(x) \leq \fint_{B(x,r)} v(y)\, dy$$

(b) Use the previous property to prove the maximum principle for subharmonic functions:

$$\max_{\bar{U}} v = \max_{\partial U} v\,.$$

Try also to give an elementary direct proof of this result.
Hint: first consider the case $-\Delta v < 0$. Then prove the general case by approximating v with the functions $v_\varepsilon(x) := v(x) + \varepsilon |x|^2$.

(c) Consider $\phi \in C^\infty(\mathbb{R})$ **convex** and let u be harmonic in U. Prove that the composition $\phi(u)$ is subharmonic.

(d) Let u be harmonic in U. Prove that $|\nabla u|^2$ is subharmonic.

Solution: (a) Fix $x \in U$ and $r > 0$ such that $B(x,r) \subset U$. Define the function

$$\psi(\rho) := \fint_{\partial B(x,\rho)} v(\xi)\, dS(\xi) = \fint_{\partial B(0,1)} v(x + \rho\zeta)\, dS(\zeta)$$

for $\rho \in (0,r)$. Observe that

$$\lim_{\rho \to 0^+} \phi(\rho) = v(x)\,. \tag{1.1}$$

Now using the divergence theorem one has

$$\psi'(\rho) = \fint_{\partial B(0,1)} \nabla v(x+\rho\zeta) \cdot \zeta\, dS(\zeta) = \fint_{\partial B(x,\rho)} \nabla v(\xi) \cdot \frac{\xi - x}{\rho}\, dS(\xi) =$$
$$= \fint_{\partial B(x,\rho)} \frac{\partial v}{\partial \nu}(\xi)\, dS(\xi) = \frac{r}{N} \fint_{B(x,\rho)} \Delta v(y)\, dy \geq 0$$

so that the function $\psi(\rho)$ is increasing. Taking into account (1.1) this gives

$$\psi(\rho) \geq \psi(0+) = v(x)$$

whence, also using Fubini's theorem, and denoting by α_N the voulume of the unit sphere

$$\fint_{B(x,r)} v(y)\, dy = \frac{1}{\alpha_N r^N} \int_{B(x,r)} v(y)\, dy = \frac{1}{\alpha_N r^N} \int_0^r \Big(\int_{\partial B(x,\rho)} v(\xi)\, dS(\xi) \Big)\, d\rho =$$
$$= \frac{1}{r^N} \int_0^r N\rho^{N-1} \psi(\rho)\, d\rho \geq \frac{v(x)}{r^N} \int_0^r N\rho^{N-1}\, d\rho = v(x)\,,$$

as required.

(b) Assume that v takes its maximum at an interior point x_0. But for every $r > 0$ such that $B(x_0, r) \subset U$ one has, by part (a)

$$\fint_{B(x_0,r)} v(y) - v(x_0)\, dy \geq 0\,;$$

since the integrand must be nonpositive this implies $v(y) = v(x_0)$ for every $y \in B(x_0, r)$. It follows that v is constant in the connected component of U containing x_0 (*strong maximum principle*) which implies the weaker conclusion. **Alternative proof will appear here next week**.

(c) By the usual rules of differentiation, one has

$$\Delta[\phi(u)] = \phi'(u)\Delta u + \phi''(u)|\nabla u|^2 .$$

Since $\Delta u = 0$ and $\phi'' \geq 0$ by convexity, one gets $\Delta[\phi(u)] \geq 0$ as required.

(d) For every $i = 1, \dots, N$ the derivatives $\frac{\partial u}{\partial x_i}$ are still harmonic. By the previous step with $\phi(t) = t^2$, the functions $\left(\frac{\partial u}{\partial x_i}\right)^2$ are subharmonic. Then

$$|\nabla u|^2 = \sum_{i=1}^{N} \left(\tfrac{\partial u}{\partial x_i}\right)^2$$

must be subharmonic as well. □

2. Exercise

Let $U \in \mathbb{R}^N$ be a bounded open set. Let $v \in C^2(\bar{U})$ satisfy

$$\begin{cases} -\Delta v = f & \text{in } U \\ v = g & \text{on } \partial U . \end{cases}$$

Prove that there exist a constant C_U only depending on U such that

$$\max_{\bar{U}} |v| \leq C_U (\max_{\partial U} |g| + \max_{\bar{U}} |f|) .$$

Solution: Let $L := \max_{\bar{U}} |f|$, and define $\hat{v}(x) := v(x) + \frac{L}{2N}|x|^2$. One has

$$-\Delta \hat{v} = f - L = f - \max_{\bar{U}} |f| \leq 0$$

and therefore by the maximum principle for subharmonic functions and trivial estimates

$$\max_{\bar{U}} v \leq \max_{\bar{U}} \hat{v} = \max_{\partial U} \hat{v} \leq \max_{\partial U} g + L \max_{y \in \partial U} \frac{1}{2N}|y|^2 .$$

Therefore, setting $M_U := \max_{y \in \partial U} \frac{1}{2N}|y|^2$ and recalling the expression of L, we get

$$\max_{\bar{U}} v \leq \max_{\partial U} g + M_U \max_{\bar{U}} |f| \leq \max_{\partial U} |g| + M_U \max_{\bar{U}} |f| \tag{2.1}$$

where the last estimate is obvious. The same argument with $-v$ in place of v gives

$$\max_{\bar{U}}(-v) \leq \max_{\partial U} |-g| + M_U \max_{\bar{U}} |-f| = \max_{\partial U} |g| + M_U \max_{\bar{U}} |f| . \tag{2.2}$$

Combining (2.1) and (2.2) gives

$$\max_{\bar{U}} |v| \leq \max_{\partial U} |g| + M_U \max_{\bar{U}} |f|$$

and eventually the conclusion, with C_U the larger between 1 and M_U. □

EXERCISES PDE WS 2012-2013

1. Exercises PDE 07.11.12-09.11.12

Exercise 1. Let $U \subset \mathbb{R}^N$ be a bounded open set. Consider a sequence u_n of harmonic functions such that $u_n \rightrightarrows u$ uniformly on U. What can we say about u?

Solution: u is a harmonic function, too. Actually, u is continuous since it is the limit of continuous functions; moreover, $x \in U$ and $r > 0$ such that $B(x,r) \subset U$ one has

$$u(x) = \lim_{n\to+\infty} u_n(x) = \lim_{n\to+\infty} \fint_{\partial B(x,r)} u_n(\xi)\, dS(\xi) = \fint_{\partial B(x,r)} u(\xi)\, dS(\xi)\,,$$

that is, it holds the mean-value property. This one implies first that $u \in C^\infty(U)$ (remember the argument with the mollifiers!), and eventually that u is harmonic, since such is a C^∞ function with the mean value property. ☐

Exercise 2. Fix $R > 0$ and consider a *nonnegative* function $u \geq 0$ which is harmonic in the ball $B(0,R)$. Prove the following **Harnack's inequality**: for every x such that $|x| < R$ one has

$$R^{N-2}\frac{R-|x|}{(R+|x|)^{N-1}}u(0) \leq u(x) \leq R^{N-2}\frac{R+|x|}{(R-|x|)^{N-1}}u(0)\,. \tag{1.1}$$

Deduce that

$$\sup_{B(0,\frac{R}{2})} u \leq 3^N \inf_{B(0,\frac{R}{2})} u\,.$$

Solution: Denote by α_N the volume of the unit sphere. By Poisson's formula, for every x such that $|x| < R$ one has

$$u(x) = \int_{\partial B(0,R)} K_R(x,\xi)\, u(\xi)\, dS(\xi)$$

with K_R the Poisson's kernel

$$K_R(x,\xi) := \frac{R^2 - |x|^2}{N\alpha_N R\, |\xi - x|^N}\,.$$

Since $|\xi| = R > |x|$ by means of elementary triangle inequalities we get

$$\frac{R-|x|}{N\alpha_N R\,(R+|x|)^{N-1}} \leq K_R(x,\xi) \leq \frac{R+|x|}{N\alpha_N R\,(R-|x|)^{N-1}}$$

and since u is nonnegative, substituting into the Poisson's formula we arrive at

$$\frac{R-|x|}{N\alpha_N R\,(R+|x|)^{N-1}}\int_{\partial B(0,R)} u(\xi)\, dS(\xi) \leq u(x) \leq \frac{R+|x|}{N\alpha_N R\,(R-|x|)^{N-1}}\int_{\partial B(0,R)} u(\xi)\, dS(\xi)\,.$$

Since by the mean-value formula

$$\int_{\partial B(0,R)} u(\xi)\, dS(\xi) = N\alpha_N R^{N-1}\, u(0)\,,$$

we eventually get (1.1).

Now, for $|x| = \frac{R}{2}$, Harnack's inequality reads, after simple computations,

$$\frac{2^{N-2}}{3^{N-1}}u(0) \leq u(x) \leq 3\cdot 2^{N-2}u(0)\,,$$

which gives

$$\max_{\partial B(0,\frac{R}{2})} u \leq 3\cdot 2^{N-2}u(0) \quad \text{and} \quad \min_{\partial B(0,\frac{R}{2})} u \geq \frac{2^{N-2}}{3^{N-1}}u(0)\,.$$

Combining the two we arrive at

$$\max_{\partial B(0,\frac{R}{2})} u \leq 3^N \min_{\partial B(0,\frac{R}{2})} u$$

which entails the conclusion, by the maximum and minimum principles. □

Exercise 3. Find an explicit solution for the following Dirichlet problem on the square $Q_L := (0,L)^2$ associated to the Laplace equation

$$\begin{cases} \Delta u(x,y) = 0 \\ u(0,y) = f(y) \\ u(L,y) = 0 \\ u(x,0) = 0 \\ u(x,L) = 0 \end{cases}$$

in the following cases:

(a) $f(y) = \sin(\frac{n\pi}{L}y)$, with $n \in \mathbb{N}$;

(b) f is a generic C^1 function satisfying $f(0) = f(L) = 0$ and thus can be developed in a Fourier series

$$f(y) = \sum_{n=1}^{+\infty} a_n \sin(\frac{n\pi}{L}y)$$

with

$$\sum_{n=1}^{+\infty} |a_n| < +\infty\,. \tag{1.2}$$

Hint: first, try to see which functions of the form $u(x,y) = v(x)w(y)$ satisfy the Laplace equation with 0 boundary conditions on three sides of the square.

Solution: We first observe that due to the boundary conditions, v and w must satisfy

$$w(0) = w(L) = 0 \tag{1.3}$$

and

$$v(L) = 0\,. \tag{1.4}$$

Moreover, the Laplace equation gives

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)}$$

for every x and y; but this is possible only if both sides are equal to a constant, and thus there must exist $\lambda \in \mathbb{R}$ such that

$$v''(x) - \lambda v(x) = w''(y) + \lambda w(y) = 0 \tag{1.5}$$

for every x and y. Taking into account the boundary conditions (1.3), the function w has to solve a **boundary-value problem** of the form

$$\begin{cases} w''(y) + \lambda w(y) = 0 \\ w(0) = w(L) = 0\,. \end{cases} \tag{1.6}$$

Without entering the details of the related (beautiful) theory, we only observe that by solving explicitly and imposing the boundary conditions, one sees that the problem (1.6) has solution only for a discrete set of values of λ (no wonder: this is not a Cauchy problem)! Precisely, it must be

$$\lambda = \lambda_n := \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N} \tag{1.7}$$

and for given n the solutions of (1.6) are all multiples of the functions

$$w_n(y) := \sin\big(\frac{n\pi}{L}y\big)\,, \quad n \in \mathbb{N}\,.$$

For fixed n, now, and thus for fixed λ_n, by (1.5) and (1.4) we can recover the functions $v_n(x)$ as solutions of the problem

$$\begin{cases} v_n''(y) - \lambda_n v_n(y) = 0 \\ v_n(L) = 0\,. \end{cases}$$

The solutions of this problem are all multiples of the functions

$$v_n(x) := \sinh\big(\frac{n\pi}{L}(x-L)\big)\,.$$

We recall that the function $\sinh(t) := \frac{1}{2}(e^t - e^{-t})$ is an **odd increasing** function, $\sinh(t) \geq 0 \iff t \geq 0$.

We can now take a small breath by observing that we have answered the question posed by the hint: the harmonic functions we searched are of the form

$$u_n(x,y) = \mu \sinh\big(\frac{n\pi}{L}(x-L)\big) \sin\big(\frac{n\pi}{L}y\big)\,, \quad \mu \in \mathbb{R}, n \in \mathbb{N}\,. \tag{1.8}$$

Now for $f(y) = \sin(\frac{n\pi}{L}y)$ we immediately get that the solution can be recovered from (1.8) by imposing the boundary condition (namely, recover μ by imposing equality for $x = 0$) and we get the solutions

$$\overline{u}_n(x,y) = -\frac{1}{\sinh(n\pi)} \sinh\big(\frac{n\pi}{L}(x-L)\big) \sin\big(\frac{n\pi}{L}y\big)\,.$$

In the general case, we first proceed formally. Observe that finite linear combinations of the u_n given by (1.8) are still harmonic and 0 on the three sides of the square; but since f is given by an infinite sum, we look for a candidate solution in the form of an infinite linear combination

$$u(x,y) = \sum_{n=1}^{+\infty} b_n \sinh\big(\frac{n\pi}{L}(x-L)\big) \sin\big(\frac{n\pi}{L}y\big)\,.$$

By imposing equality for $x = 0$ with the Fourier development of f we get that our candidate solution is

$$u(x,y) = \sum_{n=1}^{+\infty} -\frac{a_n}{\sinh(n\pi)} \sinh\big(\frac{n\pi}{L}(x-L)\big) \sin\big(\frac{n\pi}{L}y\big)\,. \tag{1.9}$$

But since the solution is given by an infinite series, we are not done. We must first show that the series converges, to have a well defined function! And actually, we want to show that it converges uniformly on $\overline{Q}_L$: thus, the limit will be a harmonic function by Exercise 1 (by construction the partial sums are harmonic functions), while the boundary conditions are attained by the choice of the coefficients. By the properties of $\sinh$, we now observe that for every $x \in [0, L]$ we have

$$0 \leq -\frac{\sinh\big(\frac{n\pi}{L}(x-L)\big)}{\sinh(n\pi)} \leq 1$$

and thus, for every $(x,y) \in \overline{Q}_L$

$$\Big| -\frac{a_n}{\sinh(n\pi)} \sinh\big(\frac{n\pi}{L}(x-L)\big) \sin\big(\frac{n\pi}{L}y\big)\Big| \leq |a_n|\,;$$

the required uniform convergence follows now by (1.2). $\square$

2. Exercises PDE 14.11.12-16.11.12

Exercise 1. *Harnack's Theorem*: consider an increasing sequence $u_n\colon B(0,R)\to\mathbb{R}$ of harmonic functions, that is $u_n(x)\le u_{n+1}(x)$ for every $x\in B(0,R)$. Assume that $u_n(0)$ is a Cauchy sequence. Then there exists a harmonic function u such that $u_n\rightrightarrows u$ uniformly in $B(0,r)$ for every $r<R$.

Solution*:* It suffices to show that $u_n(x)$ is a Cauchy sequence uniformly for $x\in B(0,r)$. To this aim, fix $h\in\mathbb{N}$, and observe that, by the hypothesis, for every $k\le h$ we have that u_k-u_h is a positive harmonic function. Thus, by Harnack's inequality there exists a constant $C(N,r,R)$ such that

$$\sup_{x\in B(0,r)}|u_k(x)-u_h(x)| = \sup_{x\in B(0,r)}(u_k(x)-u_h(x))$$
$$\le C(N,r,R)\inf_{x\in B(0,r)}(u_k(x)-u_h(x)) \le C(N,r,R)(u_k(0)-u_h(0)).$$

Since the last term of this chain of inequalities is by the hypothesis a Cauchy sequence, we get the conclusion. □

Exercise 2. Let $u\colon\mathbb{R}^N\to\mathbb{R}$ be a harmonic function. Prove that:

(a) $\displaystyle\int_{\mathbb{R}^n}u^2(x)\,dx<+\infty\Rightarrow u\equiv 0;$

(b) $\displaystyle\int_{\mathbb{R}^n}|\nabla u|^2(x)\,dx<+\infty\Rightarrow u\equiv \text{const}.$

Solution*:* For part (a), observe that u^2 is subharmonic (Ex.1 (c), 31.10.12). Then by the mean-value property one has, for every $x\in\mathbb{R}^n$ and every $R>0$

$$u^2(x)\le\frac{1}{\alpha_N R^N}\int_{B(x,R)}u^2(x)\,dx\le\frac{1}{\alpha_N R^N}\int_{\mathbb{R}^n}u^2(x)\,dx=\frac{C}{\alpha_N R^N}.$$

Letting R to $+\infty$, we get $u^2(x)\le 0$ for every x, which implies the conclusion.

For part (b), apply part (a) to the harmonic functions $\frac{\partial}{\partial x_i}u$ with $i=1,\dots,N$ to obtain $\nabla u\equiv 0$ and thus the conclusion. □

Exercise 3.

- Consider a holomorphic function $f\colon\mathbb{C}\to\mathbb{C}$ and its real and imaginary part $u\colon\mathbb{R}^2\to\mathbb{R}$ and $v\colon\mathbb{R}^2\to\mathbb{R}$, that is
$$f(z)=u(x,y)+i\,v(x,y) \tag{2.1}$$
with $z=x+iy$. Prove that u and v are harmonic.
- Prove the following converse. Let $u\colon B(0,R)\subset\mathbb{R}^2\to\mathbb{R}$ be a harmonic function. Then, there exists $v\colon B(0,R)\subset\mathbb{R}^2\to\mathbb{R}$ harmonic such that $f(z)$ defined by (2.1) is a holomorphic function. Is v unique?

Solution*:* We need to recall the following preliminaries about holomorphic functions. By the Cauchy-Riemann theorem, f is holomorphic in region of the complex plane D if and only if the following conditions are satisfied

$$\begin{cases}\dfrac{\partial}{\partial x}u=\dfrac{\partial}{\partial y}v\\[2mm] \dfrac{\partial}{\partial y}u=-\dfrac{\partial}{\partial x}v\end{cases} \tag{2.2}$$

and if this happens one has

$$f'(z_0)=\frac{\partial}{\partial x}u(x_0,y_0)+i\frac{\partial}{\partial x}v(x_0,y_0) \tag{2.3}$$

for every $z_0\in D$, with $z_0=x_0+iy_0$.

It also follows that if Γ is a Jordan curve surrounding a domain D and f is holomorphic in D, then the line integral of f over Γ vanishes: in formulas

$$\int_\Gamma f(z)\,dz=0 \tag{2.4}$$

Indeed, by (2.1), $dz = dx + i\,dy$ we get

$$\int_\Gamma f(z)\,dz = \int_\Gamma P(x,y)\cdot ds + i\int_\Gamma Q(x,y)\cdot ds\,,$$

where the vector fields P and Q are given by $P = (u,-v)$ and $Q = (v,u)$ respectively; by Stokes' Theorem and (2.2) we get

$$\int_\Gamma P(x,y)\cdot ds = \int_D \mathrm{curl}P(x,y)\,dxdy = 0$$
$$\int_\Gamma Q(x,y)\cdot ds = \int_D \mathrm{curl}Q(x,y)\,dxdy = 0\,,$$

proving (2.4). With these preliminaries we can solve the exercise.

First, implication (a) is quite easy. Indeed, if f is holomorphic, by (2.2) we get

$$\Delta u = \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u = \frac{\partial^2}{\partial x\partial y}v - \frac{\partial^2}{\partial y\partial x}v = 0$$

since we can exchange the order or derivations by smoothness of u. A similar proof shows that v is harmonic, too.

For part (b), we start by observing that if f holomorphic having u as its real part exists, then by (2.3) and (2.2) it must be

$$f'(z) = \frac{\partial}{\partial x}u(x,y) - i\frac{\partial}{\partial y}u\,. \tag{2.5}$$

Then, defining $g(z)$ as the right-hand side, which depends only on u, it only suffices to show that there exists a holomorphic primitive f of g satisfying $f(0) = u(0)$. If it exists, its imaginary part v will be harmonic by part (a), and uniquely determined up to a constant, thus concluding the exercise.

By analogy with integral calculus in one variable, we set

$$f(z) = u(0) + \int_{[0,z]} g(w)\,dw$$

where $[0,z]$ is the line segment connecting 0 and z, oriented from 0 to z. We observe that g satisfies the Cauchy-Riemann conditions; indeed, since u is smooth and harmonic

$$\begin{cases} \frac{\partial}{\partial x}\mathrm{Re}g = \frac{\partial^2}{\partial x^2}u = -\frac{\partial^2}{\partial y^2}u = \frac{\partial}{\partial y}\mathrm{Im}g \\ \frac{\partial}{\partial x}\mathrm{Im}g = -\frac{\partial^2}{\partial x\partial y}u = -\frac{\partial^2}{\partial y\partial x}u = -\frac{\partial}{\partial x}\mathrm{Re}g\,. \end{cases}$$

From this, by (2.4) with Γ the union of the three segments $[0,z_0]$, $[z_0,z]$ and $[z,0]$ all oriented from the first to the second endpoint (observe that this curve is the boundary of a domain interely contained in the ball, thus g is holomorphic inside!) we easily get

$$\int_{[0,z]} g(w)\,dw - \int_{[0,z_0]} g(w)\,dw = \int_{[z_0,z]} g(w)\,dw\,,$$

so that

$$\frac{f(z)-f(z_0)}{z-z_0} = \frac{\int_{[z_0,z]} g(w)\,dw}{z-z_0}$$

for every z and z_0 in $B(0,R)$; and now, only continuity of g suffices to say that the righ-hand side has a limit when z goes to z_0, which is equal to $g(z_0)$, as we wanted. □

Remark 2.1. In Exercise 3 part (a) it was used the fact that when f is holomorphic, then u and v are C^∞. Actually, this is a consequence of the fact that the derivative of a holomorphic function is still holomorphic, usually recovered as a consequence of Cauchy's formula for the derivatives. It is interesting to show that this fact can be also derived **only using** the Cauchy-Riemann conditions and the theory of harmonic functions presented so far. I give a sketch of the proof.

Let f, u and v as in (2.1) and consider $f_n = \rho_n \star u + i(\rho_n \star v)$, where ρ_n are the usual mollifiers. It is easy to see that $\rho_n \star u$ and $\rho_n \star v$ (which are now C^∞) still satisfy (2.2) (why?) so that f_n are holomorphic and $\rho_n \star u$ and $\rho_n \star v$ harmonic. As $\rho_n \star u$ and $\rho_n \star v$ converge locally uniformly to u and v with n going to $+\infty$, we get that u and v are harmonic, and in particular C^∞. Now, (2.5) holds, and since u is harmonic we get (how?) that f' is holomorphic, as required.

3. Exercises PDE 21.11.12-23.11.12

Exercise 1. Assume $N = 1$ and $u(x,t) = v(\frac{x}{\sqrt{t}})$.

(a) Show that $u_t = u_{xx}$ if and only if v solves the differential equation
$$v"(z) + \frac{z}{2}v'(z) = 0$$
and compute the solutions of it in dependance of two arbitrary constants c and d.

(b) Differentiate with respect to x and select the constant c properly, to obtain the fundamental solution for $N = 1$. Why this procedure gives the fundamental solution? (*Hint:* What is the initial condition for u?)

Solution: (a) It holds, denoting with primes the derivatives of v with respect to its argument z:
$$\frac{\partial}{\partial t}u(x,t) = \frac{-x}{2t\sqrt{t}}v'(z) = -\frac{zv'(z)}{2t}$$
$$\frac{\partial^2}{\partial x^2}u(x,t) = \frac{1}{2t}v''(z)$$
so that $u_t = u_{xx}$ if and only if v solves the differential equation
$$v"(z) + \frac{z}{2}v'(z) = 0\,.$$

(b) Setting $w = v'$ one has that it must be
$$w'(z) = ce^{\frac{-z^2}{4}}$$
for some arbitrary constant $c \in \mathbb{R}$, whence
$$v(z) = c\int_0^z e^{\frac{-s^2}{4}}\,ds + d$$
with $d \in \mathbb{R}$ again arbitrary. It is easy to check that
$$U(x,t) := \frac{\partial}{\partial x}u(x,t) = \frac{c}{\sqrt{t}}e^{\frac{-x^2}{4t}}$$
solves again the heat equation. Moreover we can impose to this function the "initial condition" for the fundamental solution, that is it must approximate the Dirac delta when t tends to 0. This results in the coupling of the two conditions
$$\lim_{t\to 0} U(x,t) = 0 \text{ if } |x| \neq 0$$
and
$$\int_{\mathbb{R}} U(x,t)\,dx = 1 \tag{3.1}$$
for t in a neighborhood of 0. The first is automatically satisfied, while the second gives the required constant $c = \frac{1}{2\sqrt{\pi}}$.

Also, observe that (3.1) asserts conservation of the total heat, a property that follows from the equation for a solution decaying at infinity with its derivative. It is actually, for $t \neq 0$
$$-\frac{d}{dt}\int_{\mathbb{R}} U(x,t)\,dx = \int_{\mathbb{R}} U_{xx}(x,t)\,dx = U_x(+\infty,t) - U_x(-\infty,t) = 0$$
which implies (3.1) possibly with a constant different from 1, provided that the total heat keeps bounded for t going to 0 (if not we have identically $+\infty$, as it happens for $u(x,t)$!) □

Exercise 2. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f & \text{in } \mathbb{R}^n \times (0,+\infty) \\ u = g & \text{in } \mathbb{R}^n \times \{t=0\} \end{cases} \tag{3.2}$$

where $c \in \mathbb{R}$.

Solution: Set $v(t,x) = e^{ct}\, u(t,x)$ where u solves (3.2). By a direct computation one gets

$$v_t - \Delta v = e^{ct}(u_t - \Delta u + cu)$$

in $\mathbb{R}^n \times (0,+\infty)$ therefore v solves

$$\begin{cases} v_t - \Delta v = e^{ct} f(x,t) & \text{in } \mathbb{R}^n \times (0,+\infty) \\ v = g & \text{in } \mathbb{R}^n \times \{t=0\}\,. \end{cases}$$

A solution of the previous problem is given via Duhamel's principle by

$$v(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^n} e^{\frac{-|x-y|^2}{4t}} g(y)\,dy + \int_0^t \int_{\mathbb{R}^n} \frac{e^{cs}}{(4\pi(t-s))^{\frac{N}{2}}} e^{\frac{-|x-y|^2}{4(t-s)}} f(y,s)\,dsdy$$

Now, setting $u(x,t) := e^{-ct} v(t,x)$, with v as above, gives a solution to (3.2). □

Exercise 3. *Tychonov's counterexample:* Consider the holomorphic function $g(z) = e^{-\frac{1}{z^2}}$ for $z \in \mathbb{C}\backslash\{0\}$ and denoting by $g^{(k)}$ its k-th derivative, define the function

$$u(x,t) = \begin{cases} \displaystyle\sum_{k=0}^{+\infty} \frac{g^{(k)}(t)}{(2k)!} x^{2k} & \text{if } t > 0\,, x \in \mathbb{R} \\ 0 & \text{if } t = 0\,, x \in \mathbb{R} \end{cases}$$

Rigorously justify that this is a solution of the heat equation with 0 Cauchy datum by showing uniform convergence of the series (and of the series of its time and space derivatives involved in the equation) on any semi-strip of the type

$$(x,t) \in [-a,a] \times [\delta,+\infty)$$

with $a, \delta > 0$.

Hint: apply Cauchy's formula for the derivatives of holomorphic functions in this form

$$g^{(k)}(t) = \frac{k!}{2\pi\, i} \int_{\partial B(t,\frac{t}{2})} \frac{g(z)}{(z-t)^{k+1}}\, dz \tag{3.3}$$

to estimate $g^{(k)}(t)$. Obviously, if you find a method not using complex analysis, it is also valid!

Solution: For every $z \in C_t$ one has that there exists $\omega \in [0,2\pi)$ such that

$$z = t(1 + \frac{1}{2} e^{i\omega})$$

whence

$$\frac{1}{z} = \frac{1}{t} \frac{1}{1+\frac{1}{2}e^{i\omega}} = \frac{1}{t} \frac{2(2+e^{-i\omega})}{5+4\cos\omega}.$$

Therefore

$$\frac{1}{z^2} = \frac{4}{t^2} \frac{(2+e^{-i\omega})^2}{(5+4\cos\omega)^2}.$$

so that

$$\mathrm{Re}\frac{1}{z^2} = \frac{4}{t^2} \frac{4+\cos(2\omega) - 4\cos(\omega)}{(5+4\cos\omega)^2} \geq \frac{4}{81t^2}\,,$$

where the last estimate is simply obtained by observing that 0 is a minimizer of the function at the numerator, plus the trivial fact that $(5+4\cos\omega)^2 \leq 81$.

With this, we get

$$|e^{-\frac{1}{z^2}}| = e^{-\mathrm{Re}\left(\frac{1}{z^2}\right)} \leq e^{-\frac{4}{81t^2}}$$

for every $z \in C_t$, therefore (3.3) gives

$$|g^{(k)}(t)| \leq \frac{k!}{2\pi} \int_{\partial B(t,\frac{t}{2})} \frac{|g(z)|}{|z-t|^{k+1}} |dz| \leq \frac{k!}{2\pi} \int_0^{2\pi} 2^{k+1} \frac{e^{-\mathrm{Re}\left(\frac{1}{z^2}\right)}}{t^{k+1}} \frac{t}{2} \, d\theta \leq k! \Big(\frac{2}{t}\Big)^k e^{-\frac{4}{81t^2}} .$$

By recalling the easy estimate

$$\frac{2^k k!}{(2k)!} \leq \frac{1}{k!}$$

for every $k \in \mathbb{N}$, we conclude that

$$\Big| \frac{g^{(k)}(t)}{(2k)!} x^{2k} \Big| \leq e^{-\frac{4}{81t^2}} \left(\frac{x^2}{t}\right)^k \frac{1}{k!}$$

for every $k \in \mathbb{N}$ and every (x,t). Therefore the series defining $u(x,t)$ is totally and thus uniformly convergent in $[-a,a] \times [\delta, +\infty)$; moreover, we have the estimate

$$|u(x,t)| \leq e^{-\frac{4}{81t^2} + \frac{a^2}{t}}$$

for every $x \in [-a,a]$ and $t > 0$. This implies

$$\lim_{t \to 0^+} |u(x,t)| = 0$$

uniformly in $x \in [-a,a]$, which means that $u \in C(\mathbb{R} \times [0,+\infty))$.

A similar proof shows that the series

$$\sum_{k=0}^{+\infty} \frac{g^{(k+1)}(t)}{(2k)!} x^{2k} , \tag{3.4}$$

formally giving $u_t(x,t)$, as well as

$$\sum_{k=1}^{+\infty} \frac{g^{(k)}(t)}{(2k-1)!} x^{2k-1}$$

and

$$\sum_{k=1}^{+\infty} \frac{g^{(k)}(t)}{(2(k-1))!} x^{2(k-1)} \tag{3.5}$$

formally giving $u_x(x,t)$, and $u_{xx}(x,t)$, respectively, uniformly converge in $[-a,a] \times [\delta,+\infty)$. Therefore $u(x,t)$ is twice differentiable in x and once t; a trivial algebraic computation using (3.4) and (3.5) shows that it solves the heat equation, as required. □

4. Exercises PDE 28.11.12-30.11.12

Exercise 1.

(a) Show that the general solution of the PDE $u_{xy} = 0$ is

$$u(x,y) = F(x) + G(y) \tag{4.1}$$

for arbitrary functions F and G.

(b) Using the change of variables $\xi = x + t$, $\eta = x - t$, show that $u_{tt} - u_{xx} = 0$ if and only if $u_{\xi\eta} = 0$.

(c) Rederive D'Alembert's formula.

(d) Under which conditions on the initial data g, h is the solution a right-moving wave? A left-moving wave?

Solution: (a) Clearly any function of the form (4.1) solves the equation, and it must be only shown the converse. Take a solution u of the PDE $u_{xy} = 0$ and, for fixed x, set $f_x(y) := u_x(x,y)$. One has that

$$\frac{d}{dy} f_x(y) = 0$$

therefore

$$f_x(y) \equiv C_x \,, \tag{4.2}$$

a constant depending only on x. We now set $f(x) = C_x$ for every x, we call $F(x)$ a primitive of f, and we observe that by definition of f_x we can rewrite (4.2) as follows: for every x and y, it holds

$$u_x(x,y) = \frac{d}{dx}F(x)\,.$$

We now set, for fixed y, $g_y(x) := u(x,y)$. We have

$$\frac{d}{dx}g_y(x) = u_x(x,y) = \frac{d}{dx}F(x)$$

for every x and y. This gives

$$g_y(x) - F(x) \equiv G_y\,,$$

a constant depending only on y. Setting $G(y) := G_y$, the previous equality gives the conclusion.

(b) Setting $\xi = x + t$, $\eta = x - t$ is equivalent to $t = \frac{1}{2}(\xi - \eta)$ and $x = \frac{1}{2}(\xi + \eta)$. Consequently, let $v(\xi,\eta) := u(\frac{1}{2}(\xi-\eta), \frac{1}{2}(\xi+\eta))$. denoting with u_t differentiation with respect to the first argument and with u_x differentiation with respect to the second argument, we have

$$\frac{\partial}{\partial \eta}v(\xi,\eta) = -\frac{1}{2}u_t\Big(\frac{1}{2}(\xi-\eta), \frac{1}{2}(\xi+\eta)\Big) + \frac{1}{2}u_x\Big(\frac{1}{2}(\xi-\eta), \frac{1}{2}(\xi+\eta)\Big)$$

and therefore

$$\begin{aligned}\frac{\partial^2}{\partial\xi\partial\eta}v(\xi,\eta) = &-\frac{1}{4}u_{tt}\Big(\frac{1}{2}(\xi-\eta), \frac{1}{2}(\xi+\eta)\Big) - \frac{1}{4}u_{xt}\Big(\frac{1}{2}(\xi-\eta), \frac{1}{2}(\xi+\eta)\Big)+\\ &+\frac{1}{4}u_{tx}\Big(\frac{1}{2}(\xi-\eta), \frac{1}{2}(\xi+\eta)\Big) + \frac{1}{4}u_{xx}\Big(\frac{1}{2}(\xi-\eta), \frac{1}{2}(\xi+\eta)\Big)\end{aligned}$$

so that $u_{tt} - u_{xx} = 0$ if and only if $v_{\xi\eta} = 0$ as we wanted.

(c) By part (a) and (b) we have, denoting with u a solution of the wave equation, and for $\xi = x+t$, $\eta = x - t$:

$$u\Big(\frac{1}{2}(\xi-\eta), \frac{1}{2}(\xi+\eta)\Big) = F(\xi) + G(\eta)$$

for arbitrary functions F and G, that is

$$u(t,x) = F(x+t) + G(x-t)\,,$$

the sum of the *left-travelling wave* $F(x+t)$ and of the *right-travelling wave* $G(x-t)$.

(d) A pure left-travelling wave corresponds to $G = \mathit{const.}$, and similarly a pure right-travelling wave corresponds to $F = \mathit{const}$. Imposing the initial conditions $g(x) = u(0,x)$ and $h(x) = u_t(0,x)$ we get that it must be

$$\begin{cases} F(x) + G(x) = g(x) \\ F'(x) - G'(x) = h(x) \end{cases}$$

for every x, that is

$$\begin{cases} F(x) + G(x) = g(x) \\ F(x) - G(x) = \displaystyle\int_0^x h(s)\,ds + \mathit{const.}. \end{cases}$$

We then have $G(x) \equiv \mathit{const.}$ if and only if $g(x) - \int_0^x h(s)\,ds \equiv \mathit{const.}$, which is to say that we have a left-traveling wave if and only if $g' = h$. Similarly we have a right-traveling wave if and only if $g' = -h$. □

Exercise 2.

(a) Let $\mathbf{E} := (E_1, E_2, E_3)$ and $\mathbf{B} := (B_1, B_2, B_3)$ be solutions of the Maxwell equations

$$\begin{cases} \mathbf{E}_t = \mathrm{curl}\mathbf{B}, \quad \mathbf{B}_t = -\mathrm{curl}\mathbf{E} \\ \mathrm{div}\,\mathbf{E} = \mathrm{div}\,\mathbf{B} = 0\,. \end{cases}$$

Show

$$\mathbf{E}_{tt} - \Delta\mathbf{E} = 0, \quad \mathbf{B}_{tt} - \Delta\mathbf{B} = 0\,.$$

(b) Assume that $\mathbf{u} := (u_1, u_2.u_3)$ solves the evolution equation of linear elasticity

$$\mathbf{u}_{tt} - \mu\Delta\mathbf{u} - (\lambda + \mu)\nabla(\mathrm{div}\,\mathbf{u}) = 0 \quad \text{in } \mathbb{R}^3 \times (0, +\infty)\,. \tag{4.3}$$

Show that $w := \mathrm{div}\,\mathbf{u}$ and $\mathbf{w} := \mathrm{curl}\,\mathbf{u}$ each solve wave equation, but with different speed of propagation.

Solution: (a) We use the following vector calculus identity:

$$\mathrm{curl}(\mathrm{curl}\mathbf{F}) = \nabla(\mathrm{div}\,\mathbf{F}) - \Delta\mathbf{F} \tag{4.4}$$

for every C^2 vector field $\mathbf{F}$. Taking into account that $\mathrm{div}\,\mathbf{E} = 0$, (4.4) yields

$$\mathrm{curl}(\mathrm{curl}\mathbf{E}) = -\Delta\mathbf{E}\,.$$

Observing that by Maxwell's equations $\mathbf{E}_{tt} = \mathrm{curl}\mathbf{B_t} = -\mathrm{curl}(\mathrm{curl}\mathbf{E})$ we get that $\mathbf{E}_{tt} - \Delta\mathbf{E} = 0$. The proof of the other case is similar.

(b) Using (4.3), one has

$$\mathbf{w}_{tt} = \mathrm{curl}\,\mathbf{u}_{tt} = \mu\,\mathrm{curl}\Delta\mathbf{u} + (\lambda + \mu)\,\mathrm{curl}(\nabla(\mathrm{div}\,\mathbf{u}))\,.$$

The curl of a gradient vector field being zero, we arrive at

$$\mathbf{w}_{tt} = \mu\,\mathrm{curl}\,\Delta\mathbf{u} = \mu\,\Delta\mathbf{w}$$

by simply exchanging the order of differentiation. This is a wave equation with speed of propagation μ.

Analogously

$$w_{tt} = \mathrm{div}\,(\mathbf{u}_{tt}) = \mu\,\mathrm{div}\,(\Delta\mathbf{u}) + (\lambda + \mu)\,\mathrm{div}\,(\nabla(\mathrm{div}\,\mathbf{u})) = \mu\,\Delta w + (\lambda + \mu)\,\mathrm{div}\,(\nabla w)$$

again exchanging the order of differentiation and recalling that $w = \mathrm{div}\,\mathbf{u}$. Since $\mathrm{div}\,(\nabla w) = \Delta w$ we get

$$w_{tt} = (\lambda + 2\mu)\,\Delta w\,,$$

that is a wave equation with speed of propagation $\lambda + 2\mu$. ☐

Exercise 3. Let V be a (complex) pre-Hilbertian space and A a linear mapping from V to V. Suppose preliminarly that the dimension of V is finite and equal to $n \in \mathbb{N}$ and suppose that there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ of V consisting of eigenvectors of A relative to the eigenvalues $(\lambda_1, \ldots, \lambda_n)$.

(a) Consider the following Cauchy problem associated to a linear ODE in V:

$$\frac{d}{dt}w = Aw, \quad w(0) = g \tag{4.5}$$

and show that

$$w(t) := \sum_{i=1}^{n} \langle g, v_i\rangle e^{\lambda_i t} v_i$$

is its only solution.

(b) Give an analogous statement for this other initial value problem

$$\frac{d^2}{dt^2}w = Aw, \quad w(0) = g, \quad \frac{d}{dt}w(0) = h\,. \tag{4.6}$$

We now want to generalise this method to an infinite-dimensional situation where we look to the solution of the wave equation with periodic boundary conditions, that is

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad (x,t) \in (0, 2\pi) \times (0, +\infty) \\ u(0,x) = g(x), \quad u_t(0,x) = h(x) \\ u(t,0) - u(t,2\pi) = u_x(t,0) - u_x(t,2\pi) = u_{xx}(t,0) - u_{xx}(t,2\pi) = 0 \text{ for every } t\,. \end{cases} \tag{4.7}$$

To this end, let

$$V := \{v \in C^2([0,2\pi],\mathbb{C}) : v(0) - v(2\pi) = v_x(0) - v_x(2\pi) = v_{xx}(0) - v_{xx}(2\pi) = 0\}\,. \tag{4.8}$$

(c) Interpret (4.7) as a linear ODE of the type (4.6) in the pre-Hilbertian space V, endowed with the usual L^2 scalar product

$$\langle u, v\rangle := \frac{1}{2\pi}\int_0^{2\pi} u(x)\overline{v(x)}\,dx\,,$$

where the operator A is given by

$$A := \frac{d^2}{dx^2} : V \subset L^2((0,2\pi)) \to L^2((0,2\pi))\,. \tag{4.9}$$

What are the initial conditions? What happens of the boundary value conditions?

(d) Show that $(e^{ikx})_{k\in\mathbb{Z}}$ is an orthonormal basis of V consisting of eigenvectors of A and compute the relative eigenvalues (*hint*: Fourier series).

(e) Consider now (4.7) with $g := x^2(x-2\pi)^2$ and $h \equiv 0$. Verify that g and h belong to V, then find a formal solution to (4.7) assuming that the statement you found in (b) can be generalised to an infinite-dimensional situation, and using the basis $(e^{ikx})_{k\in\mathbb{Z}}$. Then show that you actually found a solution, in this way!

Solution: (a) Uniqueness is a well-known fact. By a direct calculation, since $\lambda_i v_i = Av_i$ we get

$$\frac{d}{dt}w(t) := \sum_{i=1}^n \langle g, v_i\rangle e^{\lambda_i t}\lambda_i v_i = \sum_{i=1}^n \langle g, v_i\rangle e^{\lambda_i t} A v_i = A\Big(\sum_{i=1}^n \langle g, v_i\rangle e^{\lambda_i t} v_i\Big)\,,$$

so that $\frac{d}{dt}w = Aw$ as we wanted. On the other hand $w(0) = \sum_{i=1}^n \langle g, v_i\rangle v_i = g$ since v_i is a orthonormal basis. In view of point (b) it is important to remark that if v_i is a basis of eigenvectors, but **no orthonormal basis**, then, denoting with g_i the components of g with respect to v_i, the solution to (4.5) is given by

$$w(t) := \sum_{i=1}^n g_i e^{\lambda_i t} v_i\,; \tag{4.10}$$

only, $g_i \neq \langle g, v_i\rangle$, in general!

(b) Setting $v(t) := \frac{d}{dt}w$ and $W := (v,w) \in V \times V$ the problem is equivalent to the following one

$$\frac{d}{dt}W = \mathbf{B}W, \quad W(0) = (g,h)$$

where the linear operator B acts on $V \times V$ as follows:

$$\mathbf{B}(w,v) := (v, Aw)\,.$$

Observe that the only hypothesis that A is diagonalisable in V does not assure that $\mathbf{B}$ is diagonalisable in $V \times V$! We have actually to distinguish two cases.

(b1) $\mathit{ker}\,A = \{0\}$: this is the case where $\mathbf{B}$ possesses a basis of eigenvectors. We make the following convention: for every $\lambda \in \mathbb{C}$ with $\lambda \neq 0$ we denote with $\sqrt{\lambda}$ the complex square root of λ with argument in $[0,\pi)$. Then, a basis in $V \times V$ made of eigenvectors of $\mathbf{B}$ is easily given by the set of vectors

$$\{(W_1)_+, (W_1)_-, \ldots, (W_n)_+, (W_n)_-\}$$

where for every $i = 1, \dots, n$ we have set

$$(W_i)_+ = \frac{1}{\sqrt{1+|\lambda_i|}}(v_i, \sqrt{\lambda_i} v_i)$$

and

$$(W_i)_- = \frac{1}{\sqrt{1+|\lambda_i|}}(v_i, -\sqrt{\lambda_i} v_i)\,.$$

The corresponding eigenvalues are $\sqrt{\lambda_i}$ and $-\sqrt{\lambda_i}$, respectively. But since this basis is no orthonormal one, we can apply case (a) only in the form (4.10), to get

$$W(t) = \sum_{i=1}^{n} (w_i)_+ e^{\sqrt{\lambda_i} t} (W_i)_+ + \sum_{i=1}^{n} (w_i)_- e^{-\sqrt{\lambda_i} t} (W_i)_-\,.$$

where $(w_i)_+$ and $(w_i)_-$ are the coefficients of $W(0)$ with respect to the basis

$$\{(W_i)_+, (W_i)_-\}_{i=1,\dots,n}$$

. Projecting on the first factor we obtain the expression of w in this form

$$w(t) = \sum_{i=1}^{n} a_i e^{\sqrt{\lambda_i} t} v_i + \sum_{i=1}^{n} b_i e^{-\sqrt{\lambda_i} t} v_i \tag{4.11}$$

with $a_i = \frac{(w_i)_+}{\sqrt{1+|\lambda_i|}}$ and $b_i = \frac{(w_i)_+}{\sqrt{1+|\lambda_i|}}$. But the key point is that now we can easily determine a_i and b_i by imposing the initial conditions, since v_i is an orthonormal basis! Precisely, it must be

$$\begin{cases} a_i + b_i = \langle g, v_i \rangle \\ \sqrt{\lambda_i} a_i - \sqrt{\lambda_i} b_i = \langle h, v_i \rangle \end{cases}$$

for every i. Solving the system gives that the coefficients a_i and b_i in (4.11) are given by

$$a_i = \frac{1}{2}\Big(\langle g, v_i \rangle + \frac{\langle h, v_i \rangle}{\sqrt{\lambda_i}}\Big), \quad b_i = \frac{1}{2}\Big(\langle g, v_i \rangle - \frac{\langle h, v_i \rangle}{\sqrt{\lambda_i}}\Big)\,. \tag{4.12}$$

(b2) $\dim\, ker\, A = m \neq 0$. Denoting with $\{v_i\}_{i=1}^{n-m}$ the eigenvectors of A relative to the nonzero eigenvalues $\{\lambda_i\}_{i=1}^{n-m}$ we see that the analogous of (4.11), that is the function

$$w_1(t) = \sum_{i=1}^{n-m} a_i e^{\sqrt{\lambda_i} t} v_i + \sum_{i=1}^{n-m} b_i e^{-\sqrt{\lambda_i} t} v_i\,, \tag{4.13}$$

wtih a_i and b_i given by (4.12) still solves the equation in (4.6), but with a different initial datum: precisely $w_1(0) = g - P_{ker\,A}\, g$ and $\frac{d}{dt} w_1(0) = h - P_{ker\,A}\, h$ where $P_{ker\,A}$ denotes the orthogonal projection onto $ker\, A$. But now, given an orthonormal basis $\{v_1^{ker}, \dots, v_m^{ker}\}$ of $ker\, A$ it is easy to check that the unique solution of (4.6) is $w(t) = w_1(t) + z(t)$ where $w_1(t)$ is given by (4.13) and (4.12) and $z(t)$ is given by:

$$z(t) := \sum_{j=1}^{m} [\langle g, v_j^{ker} \rangle + \langle h, v_j^{ker} \rangle t] v_j^{ker}\,. \tag{4.14}$$

Indeed $z(t) \in ker\, A$ for every t, $\frac{d^2}{dt^2} z(t) \equiv 0 = Az(t)$, and $z(0) = P_{ker\,A}\, g$ as well as $\frac{d}{dt} z(0) = P_{ker\,A}\, h$ so that we conclude by linearity.

(c) Simply interpret (4.7) as a generalised Cauchy problem

$$\frac{d^2}{dt^2} w = Aw, \quad w(0) = g, \quad \frac{d}{dt} w(0) = h\,.$$

with $w : \mathbb{R} \to V$, where this one is given by (4.8), and A is given by (4.9). Setting $u(t, \cdot) = w(t)$ the boundary conditions ar already incorporated in the request that $w(t)$ must belong to V. Obviously, for the problem to have sense, it must be g and h in V, too!

(d) It is a very-well known fact that $(e^{ikx})_{k\in\mathbb{Z}}$ is an orthonormal basis of L^2 by the theory of Fourier series; since $(e^{ikx})_{k\in\mathbb{Z}} \subset V$ for, it constitutes an orthonormal basis of V, too. A direct calculation shows that for every $k \in Z$

$$A(e^{ikx}) = -k^2 e^{ikx}$$

so that they are eigenvectors of A with eigenvalues $-k^2$.

(e) Checking that g belongs to V is straightforward. Now, for $k \neq 0$, we have according to our convention $\sqrt{\lambda}_k = i|k|$, so that, setting

$$\alpha_k := \langle g, e^{ikx} \rangle$$

we get that the coefficients a_k and b_k in (4.12) satisfy $a_k = b_k = \frac{1}{2}\alpha_k$. Therefore the function $w_1(t)$ defined in (4.13) reduces in our case to

$$w_1(t) = \sum_{k\in\mathbb{Z},\, k\neq 0} \frac{1}{2}\alpha_k [e^{i|k|t} + e^{-i|k|t}] e^{ikx} = \sum_{k\in\mathbb{Z},\, k\neq 0} \alpha_k \cos(|k|t) e^{ikx}.$$

By a direct computation, $\alpha_k = -\frac{24}{k^4}$ for every $k \neq 0$, so that

$$w_1(t) = -\sum_{k\in\mathbb{Z},\, k\neq 0} \frac{24}{k^4} \cos(|k|t) e^{ikx} = -48 \sum_{k=1}^{\infty} \frac{1}{k^4} \cos(kt)\cos(kx).$$

The eigenvector corresponding to the eigenvalue 0 is the constant 1, so that in our case

$$z(t) = \langle g, 1 \rangle = \frac{8\pi^4}{15}$$

therefore we get to our candidate solution

$$w(t) = \frac{8\pi^4}{15} - 48 \sum_{k=1}^{\infty} \frac{1}{k^4} \cos(kt)\cos(kx). \tag{4.15}$$

The verification that (4.15) is actually the solution of (4.7) is an easy exercise in differentiation of series whose proof I omit.

□

5. Exercises PDE 5.12.12-7.12.12

Exercise 1. Let $u(t,x)\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a solution of the wave equation $u_{tt} - u_{xx} = 0$. Show that

$$v(t,x) := \int_{-\infty}^{+\infty} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{t}} u(s,x)\, ds$$

satisfies the heat equation $u_t - u_{xx} = 0$ for every $t > 0$ and $x \in \mathbb{R}$.

Solution: By D'Alembert's formula, $u(s,x) = F(x+s) + G(x-s)$ for two functions of a real variable F and G. With the change of variable $z = x + s$ one has

$$\int_{-\infty}^{+\infty} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{t}} F(x+s)\, ds = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-z)^2}{4t}}}{\sqrt{t}} F(z)\, dz$$

and similarly, with $z = x - s$ we get

$$\int_{-\infty}^{+\infty} \frac{e^{-\frac{s^2}{4t}}}{\sqrt{t}} G(x-s)\, ds = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-z)^2}{4t}}}{\sqrt{t}} G(z)\, dz$$

so that

$$v(t,x) = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-z)^2}{4t}}}{\sqrt{t}} (F(z) + G(z))\, dz$$

solving the heat equation with initial datum $2\sqrt{\pi}u(0,x)$.

□

Exercise 2. We say that a function $u\colon [0,1] \to \mathbb{R}$ is absolutely continuous if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 \le t_0 < s_0 < t_1 < s_1 < \dots t_n < s_n \le 1$ are points in $[0,1]$ satisfyng

$$\sum_{i=1}^{n} (s_i - t_i) \le \delta$$

then

$$\sum_{i=1}^{n} |u(t_i) - u(s_i)| \le \varepsilon$$

An absolutely continuous function u is continuous and $\dot{u}(t)$ (the pointwise derivative) exists for almost every $t \in [0,1]$.

Show that if $u \in W^{1,p}(0,1)$ for $1 \le p$ then there exists v absolutely continuous such that $u = v$ almost everywhere; moreover the weak derivative of u is given exactly by the pointwise derivative $\dot{v}$ and $\dot{v} \in L^p(0,1)$. Moreover, if $p > 1$:

$$|u(x) - u(y)| \le |x-y|^{1-\frac{1}{p}} (\int_0^1 |u'(t)|^p \, dt)^{\frac{1}{p}} \tag{5.1}$$

for every x and y in $[0,1]$.

Solution: The key of the exercise, important in itself, is that u satisfies the fundamental Theorem of calculus, that is

$$u(x) = u(0) + \int_0^x u'(t)\, dt \tag{5.2}$$

for every $x \in [0,1]$, with u' the weak derivative. Observe that the right-hand side is well defined, if we assume that the weak derivative belongs to L^p. To prove (5.2), no knowledge of absolutely continuous function is needed, but only Fubini's theorem and exercise 4 below. Let us see how. Given a function $v \in L^1([0,1])$, define $V(x) := \int_0^x v(y)\, dy$ and we claim that

$$V' = v\,, \tag{5.3}$$

that is, the weak derivative of V is v. To this aim, take $\varphi \in C_c^\infty([0,1])$ and observe that by Fubini's Theorem

$$-\int_0^1 V(x)\varphi'(x)\, dx = -\int_0^1 \Big(\int_0^x v(y)\, dy\Big)\varphi'(x)\, dx = -\int_0^1 \Big(\int_y^1 \varphi'(x)\, dx\Big) v(y)\, dy = \int_0^1 v(y)\varphi(y)\, dy\,.$$

This proves (5.3). Now, if you define $U(x) := u(0) + \int_0^x u'(y)\, dy$, (5.3) gives that $(U-u)' = 0$, so that by Exercise 4 below it must be $U(x) - u(x) \equiv \mathit{const.}$ for every $x \in [0,1]$. But since $U(0) = u(0)$, the constant is 0 and (5.2) is proved.

Let us also observe that (5.1) is an immediate consequence of (5.2) and the Holder's (or Jensen's inequality), when $p > 1$; it is indeed for every $0 \le x < y \le 1$

$$|u(y) - u(x)| = \Big|\int_x^y u'(t)\, dt\Big| = |y-x|\Big(\Big|\frac{1}{|y-x|}\int_x^y u'(t)\, dt\Big|^p\Big)^{\frac{1}{p}} \le |y-x|^{1-\frac{1}{p}} (\int_0^1 |u'(t)|^p \, dt)^{\frac{1}{p}}\,.$$

Absolute continuity of u follows from (5.2) with the only observation that for every $v \in L^1([0,1])$, $V(x) := \int_0^x v(y)\, dy$ is absolutely continuous. To show this, we can suppose without loss of generality that $v \ge 0$. For fixed $\varepsilon > 0$, there exist $N \in \mathbb{N}$, depending only on ε, such that defining $v_N(x) := \min\{v(x), N\}$, one has

$$\int_0^1 |v(x) - v_N(x)|\, dx \le \frac{\varepsilon}{2}\,. \tag{5.4}$$

This is because the sequence $v_k(x) := \min\{v(x), k\}$ is L^1 convergent to v. Now, fix $\delta = \frac{\varepsilon}{2N}$; for every finite sequence of points $0 \le t_0 < s_0 < t_1 < s_1 < \dots t_n < s_n \le 1$ satisfyng

$$\sum_{i=1}^{n} (s_i - t_i) \le \delta$$

one has

$$\sum_{i=1}^{n} |V(t_i) - V(s_i)| = \int_{B_n} v(y)\, dy$$

with $B_n := \cup_{i=0}^{n} [t_i, s_i]$. Since now, using (5.4) and our choice of δ

$$\int_{B_n} v(y)\, dy \le \int_{B_n} |v(y) - v_N(y)|\, dy + \int_{B_n} v_N(y)\, dy \le \frac{\varepsilon}{2} + N|B_n| \le \frac{\varepsilon}{2} + N\delta = \varepsilon\,,$$

our claim is proved.

Finally, define

$$U_h(t) := \frac{1}{h}(u(t+h) - u(t))\,.$$

By (5.2) and the Lebesgue differentiation Theorem we have that

$$U_h(t) = \frac{1}{h}\int_t^{t+h} u'(y)\, dy \to u'(t)$$

in $L^1([0,1])$ when $h \to 0$, and therefore almost everywhere. Being $U_h(t)$ exactly the difference quotients of u at t, we have shown that u is pointwise almost everywhere differentiable, and that its pointwise derivative coincides a.e. with u'. □

Exercise 3. Let $U := (-1,1) \times (-1,1)$. Define u as follows

$$u(x_1, x_2) := \begin{cases} 1 - x_1 & \text{in } T_1 := \{x_1 > 0,\, |x_2| < x_1\} \\ 1 + x_1 & \text{in } T_2 := \{x_1 < 0,\, |x_2| < -x_1\} \\ 1 - x_2 & \text{in } T_3 := \{x_2 > 0,\, |x_1| < x_2\} \\ 1 + x_2 & \text{in } T_4 := \{x_2 < 0,\, |x_1| < -x_2\} \end{cases}$$

For which $1 \le p \le \infty$ does u belong to $W^{1,p}(U)$?

Solution: Let us take $\varphi := (\varphi_1, \varphi_2) \in C_c^\infty(U; \mathbb{R}^2)$ and define the vectors $\nu_+ := (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ (unit normal to the line $\{x_1 = x_2\}$) and $\nu_- := (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ (unit normal to the line $\{x_1 = -x_2\}$). First, we compute

$$\int_{T_1} x_1 \operatorname{div} \varphi(x_1, x_2)\, dx_1 dx_2\,.$$

Since $\operatorname{div}(x_1\varphi) - (1,0)\cdot\varphi = x_1 \operatorname{div}\varphi$, by the divergence theorem (taking into account the orientation of the exterior normal!) we get

$$\begin{aligned} \int_{T_1} x_1 \operatorname{div} \varphi(x_1, x_2)\, dx_1 dx_2 = -\int_{T_1} (1,0)\, \cdot \varphi(x_1,x_2)\, dx_1 dx_2 + \\ \int_{\{x_1 = x_2;\, x_1 > 0\}} \xi_1\, \varphi(\xi)\cdot\nu_+\, ds(\xi) - \int_{\{x_1 = -x_2;\, x_1 > 0\}} \xi_1\, \varphi(\xi)\cdot\nu_-\, ds(\xi)\,. \end{aligned} \tag{5.5}$$

Here we took into account that $\varphi(1, x_2) \equiv (0,0)$. In a similar way we get

$$\begin{aligned} \int_{T_4} x_2 \operatorname{div} \varphi(x_1, x_2)\, dx_1 dx_2 = -\int_{T_4} (0,1)\, \cdot \varphi(x_1,x_2)\, dx_1 dx_2 + \\ \int_{\{x_1 = -x_2;\, x_2 < 0\}} \xi_2\, \varphi(\xi)\cdot\nu_-\, ds(\xi) - \int_{\{x_1 = x_2;\, x_2 < 0\}} \xi_2\, \varphi(\xi)\cdot\nu_+\, ds(\xi)\,. \end{aligned}$$

But this is clearly equivalent to

$$\begin{aligned} \int_{T_4} x_2 \operatorname{div} \varphi(x_1, x_2)\, dx_1 dx_2 = -\int_{T_4} (0,1)\, \cdot \varphi(x_1,x_2)\, dx_1 dx_2 + \\ -\int_{\{x_1 = -x_2;\, x_1 > 0\}} \xi_1\, \varphi(\xi)\cdot\nu_-\, ds(\xi) + \int_{\{x_1 = x_2;\, x_1 < 0\}} \xi_1\, \varphi(\xi)\cdot\nu_+\, ds(\xi)\,. \end{aligned} \tag{5.6}$$

The same reasonings lead to

$$\int_{T_2} x_1 \operatorname{div} \varphi(x_1,x_2)\,dx_1dx_2 = -\int_{T_2} (1,0)\;\cdot \varphi(x_1,x_2)\,dx_1dx_2 + \int_{\{x_1=-x_2;\,x_1<0\}} \xi_1\,\varphi(\xi)\cdot \nu_-\,ds(\xi) - \int_{\{x_1=x_2;\,x_1<0\}} \xi_1\,\varphi(\xi)\cdot\nu_+\,ds(\xi) \tag{5.7}$$

and

$$\int_{T_3} x_2 \operatorname{div} \varphi(x_1,x_2)\,dx_1dx_2 = -\int_{T_3} (0,1)\;\cdot \varphi(x_1,x_2)\,dx_1dx_2 + \int_{\{x_1=-x_2;\,x_1<0\}} \xi_1\,\varphi(\xi)\cdot \nu_-\,ds(\xi) - \int_{\{x_1=x_2;\,x_1>0\}} \xi_1\,\varphi(\xi)\cdot\nu_+\,ds(\xi) \tag{5.8}$$

Since

$$\int_U \operatorname{div} \varphi(x_1,x_2)\,dx_1dx_2 = 0$$

again by the divergence Theorem, we have that

$$-\int_U u \operatorname{div}\varphi(x_1,x_2)\,dx_1dx_2 = \int_{T_1} x_1 \operatorname{div}\varphi(x_1,x_2)\,dx_1dx_2 - \int_{T_2} x_1 \operatorname{div}\varphi(x_1,x_2)\,dx_1dx_2 + \int_{T_3} x_2 \operatorname{div}\varphi(x_1,x_2)\,dx_1dx_2 - \int_{T_4} x_2 \operatorname{div}\varphi(x_1,x_2)\,dx_1dx_2\,.$$

Using (5.5), (5.7), (5.8), and (5.6) we conclude that

$$-\int_U u \operatorname{div}\varphi(x_1,x_2)\,dx_1dx_2 = \int_U v\cdot\varphi(x_1,x_2)\,dx_1dx_2$$

where $v \in L^\infty(U;\mathbb{R}^2)$ is defined as

$$v(x_1,x_2) := \begin{cases} (-1,0) & \text{in } T_1 \\ (1,0) & \text{in } T_2 \\ (0,-1) & \text{in } T_3 \\ (0,1) & \text{in } T_4\,. \end{cases}$$

Therefore $u \in W^{1,p}(U)$ for every p. □

Exercise 4. Suppose U is connected and $u \in W^{1,p}(U)$ satisfies $Du = 0$ almost everywhere in U, with Du the Sobolev gradient. Prove that u is (almost everywhere) constant in U.

Solution: Fix a relatively compact open subset U' of U; it suffices to show u constant in U'. For ε sufficiently small, now, the convolutions with the simmetric mollifiers ρ_ε are well defined; furthermore, for every $\varphi \in C^\infty(U';\mathbb{R}^n)$, one has, by simmetry of mollifiers and since $\rho_\varepsilon \star \varphi \in C^\infty(U;\mathbb{R}^n)$ for ε small enough, that

$$\int_{U'} \operatorname{div}\varphi(x)(\rho_\varepsilon \star u)(x)\,dx = \int_U u(x)(\rho_\varepsilon \star \operatorname{div}\varphi)(x)\,dx = \int_U u(x)\operatorname{div}(\rho_\varepsilon\star\varphi)(x)\,dx\,.$$

By the hypothesis, the right-hand side is 0, so that $D(\rho_\varepsilon \star u) = 0$ on U'. By smoothness of $\rho_\varepsilon \star u$ this implies that $\rho_\varepsilon \star u$ is constant in U'; by L^p convergence, when ε is going to 0 we get that u is almost everywhere equal to a constant, as required. □

6. Exercises PDE 12.12.12-14.12.12

Exercise 1. (a) Let Ω be an open bounded subset of $\mathbb{R}^n$. For every $u \in C_c^\infty(\Omega)$, prove that the following interpolation inequality holds:

$$\Big(\int_\Omega |Du(x)|^2\,dx\Big)^2 \le C\Big(\int_\Omega u(x)^2\,dx\Big)\Big(\int_\Omega |D^2u(x)|^2\,dx\Big)\,, \tag{6.1}$$

with D^2u the Hessian matrix.

(b) Assume $\partial\Omega \in C^1$, let $u \in W^{2,2}(\Omega)\cap W_0^{1,2}(\Omega)$. Prove that u satisfies (6.1).

Solution: (a) For every $v \in C^\infty(\Omega)$, it holds

$$v\Delta v + |Dv|^2 = \operatorname{div}(v \cdot Dv)\,. \tag{6.2}$$

For $v = u$, integrating on Ω, by the divergence theorem we get

$$\int_\Omega |Du(x)|^2\,dx = -\int_\Omega u(x)\Delta u(x)\,dx$$

since $u = 0$ on $\partial\Omega$. By the Cauchy-Schwarz inequality we then have

$$\Big(\int_\Omega |Du(x)|^2\,dx\Big)^2 \le \Big(\int_\Omega u(x)^2\,dx\Big)\Big(\int_\Omega |\Delta u(x)|^2\,dx\Big) \le C\Big(\int_\Omega u(x)^2\,dx\Big)\Big(\int_\Omega |D^2u(x)|^2\,dx\Big)\,,$$

where the last estimate is obvious. This proves (6.1).

(b) By (6.2) and the divergence Theorem, we get

$$\int_\Omega |Dv_n(x)|^2\,dx = -\int_\Omega v_n(x)\Delta v_n(x)\,dx + \int_{\partial\Omega} v_n(\xi)\frac{\partial}{\partial\nu}v_n(\xi)\,dS(\xi)\,. \tag{6.3}$$

Now, considering w_n as in the hint, we obviously have

$$\int_{\partial\Omega} v_n(\xi)\frac{\partial}{\partial\nu}v_n(\xi)\,dS(\xi) = \int_{\partial\Omega}(v_n - w_n)(\xi)\frac{\partial}{\partial\nu}v_n(\xi)\,dS(\xi)$$

since $w_n(\xi) \equiv 0$. By the divergence Theorem, we then get

$$\int_{\partial\Omega} v_n(\xi)\frac{\partial}{\partial\nu}v_n(\xi)\,dS(\xi) = \int_\Omega \operatorname{div}\left[(v_n(x) - w_n(x))Dv_n(x)\right]dx =$$
$$\int_\Omega (v_n(x) - w_n(x))\Delta v_n(x)\,dx + \int_\Omega D(v_n(x) - w_n(x))\cdot Dv_n(x)\,dx\,.$$

Now, by $W^{2,2}$ convergence of v_n to u, Δv_n and Dv_n are bounded in L^2, while $\|v_n - w_n\|_{W^{1,2}(\Omega)} \to \|u - u\|_{W^{1,2}(\Omega)} = 0$ as n goes to $+\infty$, therefore

$$\lim_{n\to+\infty}\int_{\partial\Omega} v_n(\xi)\frac{\partial}{\partial\nu}v_n(\xi)\,dS(\xi) = 0\,. \tag{6.4}$$

By (6.4) and the $W^{2,2}$ convergence of v_n to u, taking the limit in (6.3) we get

$$\int_\Omega |Du(x)|^2\,dx = -\int_\Omega u(x)\Delta u(x)\,dx$$

and we conclude as in part (a). □

Exercise 2. *Poincaré's inequality:*

- Let $u \in W_0^{1,2}((0,d))$. Prove that

$$\int_0^d u(x)^2\,dx \le d^2\int_0^d u'(x)^2\,dx\,. \tag{6.5}$$

- Let $u \in W_0^{1,2}((0,d)\times\mathbb{R}^{N-1})$. Prove that

$$\int_{[0,d]\times\mathbb{R}^{N-1}} u(x)^2\,dx \le d^2\int_{[0,d]\times\mathbb{R}^{N-1}} |Du(x)|^2\,dx\,. \tag{6.6}$$

  Hint: Use density of C_c^∞ functions.
- Does the inequality hold for $u \in W^{1,2}((0,d)\times\mathbb{R}^{N-1})$?

Solution: Let x and $y \in [0,d]$. We have already seen (where?) that

$$|u(x) - u(y)| \le |x-y|^{\frac{1}{2}}\Big(\int_0^d |u'(t)|^p\,dt\Big)^{\frac{1}{2}}$$

and that u is absolutely continuous (in particular, continuous). Therefore $u \in W_0^{1,2}([0,d])$ implies $u(0)=0$ and then for $y=0$ we have

$$|u(x)| \le |x|^{\frac{1}{2}}(\int_0^d |u'(t)|^p\,dt)^{\frac{1}{2}} \le d^{\frac{1}{2}}(\int_0^d |u'(t)|^p\,dt)^{\frac{1}{2}}\,,$$

where the last estimate is trivial. Taking squares and integrating in $[0,d]$ with respect to x, we get (6.5).

Now, let $u \in C_c^\infty((0,d)\times\mathbb{R}^{N-1})$. Writing $x=(x_1,\hat{x})$ for every $x \in [0,d]\times\mathbb{R}^{N-1}$, with $x_1 \in [0,d]$ and $\hat{x} \in \mathbb{R}^{N-1}$, we have that for fixed $\hat{x}$ the function $u(\cdot,\hat{x})$ belongs to $C_c^\infty((0,d))$. We therefore have, by (6.5), that

$$\int_0^d u(x_1,\hat{x})^2\,dx_1 \le d^2\int_0^d \Big(\frac{d}{dx_1}u(x_1,\hat{x})\Big)^2\,dx_1 = d^2\int_0^d \Big(\frac{\partial}{\partial x_1}u(x)\Big)^2\,dx_1$$

for every $\hat{x} \in \mathbb{R}^{N-1}$. Now, by Fubini's Theorem

$$\int_{[0,d]\times\mathbb{R}^{N-1}} u(x)^2\,dx = \int_{\mathbb{R}^{N-1}}\Big(\int_0^d u(x_1,\hat{x})^2\,dx_1\Big)\,d\hat{x} \le$$
$$d^2\int_{\mathbb{R}^{N-1}}\Big(\int_0^d\Big(\frac{\partial}{\partial x_1}u(x)\Big)^2\,dx_1\Big)\,d\hat{x} \le d^2\int_{[0,d]\times\mathbb{R}^{N-1}}|Du(x)|^2\,dx\,.$$

This proves (6.6) when $u \in C_c^\infty((0,d)\times\mathbb{R}^{N-1})$. In the general case, there exists $u_n \in C_c^\infty(\Omega)$ such that $u_n \to u$ in $W^{1,2}(\Omega)$. In particular

$$\lim_{n\to\infty}\int_{[0,d]\times\mathbb{R}^{N-1}} u_n(x)^2\,dx = \int_{[0,d]\times\mathbb{R}^{N-1}} u(x)^2\,dx$$

and

$$\lim_{n\to\infty}\int_{[0,d]\times\mathbb{R}^{N-1}} |Du_n(x)|^2\,dx = \int_{[0,d]\times\mathbb{R}^{N-1}} |Du(x)|^2\,dx\,.$$

Since

$$\int_{[0,d]\times\mathbb{R}^{N-1}} u_n(x)^2\,dx \le d^2\int_{[0,d]\times\mathbb{R}^{N-1}} |Du_n(x)|^2\,dx\,,$$

(6.6) follows by simply taking the limit.

Such an inequality cannot hold in general in $W^{1,2}(\Omega)$. Constant functions in dimension $N=1$ provide an easy counterexample. □

Exercise 3. Let Ω be an open subset of $\mathbb{R}^n$. We denote by $\overline{C^\infty(\Omega)}^{W^{1,\infty}(\Omega)}$ the set of functions v in $W^{1,\infty}(\Omega)$ such that there exists a sequence v_n of $C^\infty(\Omega)$-functions converging to v in the $W^{1,\infty}$ norm on compact subsets of Ω.

- $\overline{C^\infty(\Omega)}^{W^{1,\infty}(\Omega)} = ?$
- In case that $\overline{C^\infty(\Omega)}^{W^{1,\infty}(\Omega)} \ne W^{1,\infty}(\Omega)$, find a function $v \in W^{1,\infty}(\Omega)$ that does not belong to $\overline{C^\infty(\Omega)}^{W^{1,\infty}(\Omega)}$.

Solution: $W^{1,\infty}$ convergence corresponds to uniform convergence of the sequence of functions and of the sequence of gradients. Therefore, since the limit of continuous functions must be continuous, one easily has that

$$\overline{C^\infty(\Omega)}^{W^{1,\infty}(\Omega)} \subseteq C^1(\Omega)\,.$$

But also the converse inclusion holds! To prove this, it simply suffices to follow exactly the steps of the Meyers-Serrin Theorem (Theorem 2, Section 5.3.2 in the book of Evans); observe actually that in step 2, since now the functions $\zeta_i u$ are $C_c^1(U)$, by mollification one can find $u_i \in C_c^\infty(\Omega)$ such that $\operatorname{supp} u_i \subset W_i$ and $\|u_i - \zeta_i u\|_{W^{1,\infty}(\Omega)} \le \frac{\delta}{2^i}$ (why?). Therefore

$$\overline{C^\infty(\Omega)}^{W^{1,\infty}(\Omega)} = C^1(\Omega)\,.$$

To find the required example, given for instance $\Omega = B(0,1)$, it suffices to find $v \in W^{1,\infty}(B(0,1))$ that does not belong to $C^1(B(0,1))$. Consider for instance $v(x) := |x|$. This is clearly no C^1 function in $B(0,1)$, since it is not differntiable at 0. On the other hand, for every $\varepsilon > 0$ one has $v \in C^1(B(0,1) \setminus \overline{B(0,\varepsilon)})$, so that, given $\varphi \in C_c^\infty(B(0,1);\mathbb{R}^n)$ one has

$$\int_{B(0,1)\setminus\overline{B(0,\varepsilon)}} v(x)\,\mathrm{div}\,\varphi(x)\,dx = -\int_{B(0,1)\setminus\overline{B(0,\varepsilon)}} \frac{x}{|x|}\cdot\varphi(x)\,dx - \int_{\partial B(0,\varepsilon)} |\xi|\,\varphi(\xi)\cdot\nu(\xi)\,dS(\xi)\,.$$

Since

$$\Big|\int_{\partial B(0,\varepsilon)} |\xi|\,\varphi(\xi)\cdot\nu(\xi)\,dS(\xi)\Big| \le N\alpha(N)\varepsilon^N\|\varphi\|_{L^\infty(B(0,1);\mathbb{R}^n)}$$

and $\frac{x}{|x|} \in L^\infty(B(0,1);\mathbb{R}^n)$, by taking the limit as $\varepsilon \to 0$ we get by dominated convergence

$$\int_{B(0,1)} v(x)\,\mathrm{div}\,\varphi(x)\,dx = -\int_{B(0,1)} \frac{x}{|x|}\cdot\varphi(x)\,dx\,,$$

which proves that $v \in W^{1,\infty}(B(0,1))$. □

7. Exercises PDE 19.12.12

Exercise 1. Let Ω be the open subset of $\mathbb{R}^N$ defined by $\Omega := B(0,1)\setminus\{x_N = 0\}$. Show that the function

$$u(x_1,x_2,\ldots,x_N) := \begin{cases} 1 & \text{if } \ x_N > 0 \\ 0 & \text{if } \ x_N < 0 \end{cases}$$

belongs to $W^{1,p}(\Omega)$ for every $1 \le p \le \infty$, but there exists no sequence v_n of $C^\infty(\overline{\Omega})$ functions such that $v_n \to u$ in $W^{1,p}(\Omega)$.

Solution: Being u a C^∞ function on each connected component of Ω, it is certainly in $W^{1,p}(\Omega)$ for every $1 \le p \le \infty$. Also observe that u is defined almost everywhere in $B(0,1)$, and actually belongs to $L^\infty(B(0,1))$, but does not belong to $W^{1,1}(B(0,1))$! Indeed, suppose by contradiction that $W^{1,1}(B(0,1))$. Then, it exists $g \in L^1(B(0,1);\mathbb{R}^n)$ such that

$$\int_{B(0,1)} u(x)\mathrm{div}\,\varphi(x)\,dx = -\int_{B(0,1)} g(x)\cdot\varphi(x)\,dx$$

for every $\varphi \in C_c^\infty(B(0,1);\mathbb{R}^n)$. Now, let us define $B(0,1)^+ := B(0,1)\cap\{x_N > 0\}$ and similarly $B(0,1)^- := B(0,1)\cap\{x_N < 0\}$. Since $Du = 0$ in $B(0,1)^+$, and obviously every $\varphi \in C_c^\infty(B(0,1)^+;\mathbb{R}^n)$ also belongs to $C_c^\infty(B(0,1);\mathbb{R}^n)$, it must be

$$\int_{B(0,1)^+} g(x)\cdot\varphi(x)\,dx = 0$$

for every $\varphi \in C_c^\infty(B(0,1)^+;\mathbb{R}^n)$. Thus, $g = 0$ a.e. in $B(0,1)^+$. Arguing similarly in $B(0,1)^-$, we get $g = 0$ a.e. in $B(0,1)$. We should therefore have

$$\int_{B(0,1)} u(x)\mathrm{div}\,\varphi(x)\,dx = 0 \tag{7.1}$$

for every $\varphi \in C_c^\infty(B(0,1);\mathbb{R}^n)$. But on the other hand, by definition of u and the divergence theorem we have

$$\int_{B(0,1)} u(x)\mathrm{div}\,\varphi(x)\,dx = \int_{B(0,1)^+} \mathrm{div}\,\varphi(x)\,dx = -\int_{B(0,1)\cap\{x_N=0\}} \varphi(\xi)\cdot e_N(\xi)\,dS(\xi)$$

for every $\varphi \in C_c^\infty(B(0,1);\mathbb{R}^n)$, and this has clearly no reason of being 0! So, $u \notin W^{1,1}(B(0,1))$.

Now, if a sequence v_n of $C^\infty(\overline{\Omega})$ functions such that $v_n \to u$ in $W^{1,p}(\Omega)$ exists, we have that v_n and Dv_n are Cauchy sequences in $L^p(\Omega)$ and $L^p(\Omega;\mathbb{R}^n)$ respectively. Since $B(0,1)$ and Ω differ by a set of Lebesgue measure 0, we also have that v_n and Dv_n are Cauchy sequences in $L^p(B(0,1))$ and $L^p(B(0,1);\mathbb{R}^n)$, respectively. Thus, v_n has a limit $\tilde{u}$ in $W^{1,p}(B(0,1))$, and it must be $\tilde{u} = u$ almost everywhere in $B(0,1)$. But since the definition of weak derivative does not depend on the Lebesgue representative, this would imply $u \in W^{1,p}(B(0,1))$, a contradiction. $\square$

Exercise 2. Let Ω be a bounded open subset of $\mathbb{R}^n$.

(a) Let $f \in W^{1,p}(\Omega)$ and $g \in W^{1,q}(\Omega)$ with $1 \le p, q \le \infty$. Find a sufficient condition on p and q such that $fg \in W^{1,1}(\Omega)$ and compute the weak gradient $D(fg)$.

(b) Show that when $N = 1$ it suffices to take $p = q = 1$.

(c) *Kettenregel*: assume that $F : \mathbb{R} \to \mathbb{R} \in C^1 \cap W^{1,\infty}(\mathbb{R})$. Show that for every $u \in W^{1,p}(\Omega)$ the composition $F(u) \in W^{1,p}(\Omega)$ and compute the weak gradient.

Solution: (a) Assume that p and q are conjugate exponents, that is $\frac{1}{p} + \frac{1}{q} = 1$. By Meyers-Serrin's Theorem, there exist two sequences f_n and g_n of $C^\infty(\Omega)$ functions such that $f_n \to f$ in $W^{1,p}(\Omega)$ and $g_n \to g$ in $W^{1,q}(\Omega)$. Now, by Holder's and Minkowsky's inequalities

$$\|f_n g_n - f_m g_m\|_1 = \|f_n(g_n - g_m) + (f_n - f_m)g_m\|_1 \le \|f_n\|_p \|g_n - g_m\|_q + \|g_m\|_q \|f_n - f_m\|_p$$

for every n and m in $\mathbb{N}$. Thus, $f_n g_n$ is a Cauchy sequence in $L^1(\Omega)$, and similarly we can prove that $D(f_n g_n) = f_n D g_n + g_n D f_n$ is a Cauchy sequence in $L^1(\Omega; \mathbb{R}^n)$. So, $f_n g_n$ is Cauchy in $W^{1,1}(\Omega)$ and is converging to fg in L^1. It follows that fg is in $W^{1,1}(\Omega)$ and that $f_n g_n$ is converging to fg in $W^{1,1}(\Omega)$. In particular,

$$D(fg) = f Dg + g Df \, .$$

(b) It suffices to consider the case of an interval, say $(0,1)$. So, let f and $g \in W^{1,1}((0,1))$. We begin with the case where f and $g \in W_0^{1,1}((0,1))$. Take f_n and g_n sequences of C_c^∞ functions converging to f, and g, respectively, in $W^{1,1}((0,1))$. Since for every $x \in [0,1]$, one has

$$|f_n(x) - f(x)| \le \int_0^1 |f_n'(y) - f'(y)| \, dy$$

we get that actually f_n converges to f uniformly in $[0,1]$, and clearly, also g_n converges to g uniformly in $[0,1]$! Thus, $f_n g_n$ converges to fg uniformly in $[0,1]$. On the other hand

$$(f_n g_n)' = f_n g_n' + g_n f_n' \, ;$$

by the uniform convergences of f_n and g_n, and the L^1 convergences of f_n' and g_n', we easily get that

$$(f_n g_n)' \to f g' + g f'$$

in $L^1((0,1))$. This proves that fg belongs to $W_0^{1,1}((0,1))$.

In the general case, we only can take f_n and g_n sequences of $C^\infty([0,1])$ functions converging to f, and g, respectively, in $W^{1,1}((0,1))$. By the fundamental Theorem of calculus, we easily get, for every m and $n \in \mathbb{N}$

$$|f_n(0) - f_m(0)| \le |f_n(x) - f_m(x)| + \int_0^1 |f_n'(y) - f_m'(y)| \, dy$$

so that integrating from 0 to 1, it is

$$|f_n(0) - f_m(0)| \le \int_0^1 |f_n(x) - f_m(x)| \, dx + \int_0^1 |f_n'(y) - f_m'(y)| \, dy$$

which yields that $f_n(0)$ is a Cauchy sequence. Now, again by the fundamental theorem of calculus, we have

$$|f_n(x) - f_m(x)| \le |f_n(0) - f_m(0)| + \int_0^1 |f_n'(y) - f_m'(y)| \, dy$$

for every $x \in (0,1)$, so that f_n is uniromly Cauchy. It follows that f_n converges to f uniformly in $[0,1]$, and similarly g_n converges to g uniformly in $[0,1]$. Arguing as in the previous step, the conclusion follows.

(c) Since $F:\mathbb{R}\to\mathbb{R}\in C^1\cap W^{1,\infty}(\mathbb{R})$, for every $u\in W^{1,p}(\Omega)$ the functions $F(u)$, $F'(u)$, and $F'(u)Du$ are a.e. well defined and belong to $L^p(\Omega)$, $L^p(\Omega)$, and $L^p(\Omega;\mathbb{R}^n)$, respectively. Now, taking a sequence u_n of $C^\infty(\Omega)$ functions such that $u_n\to u$ in $W^{1,p}(\Omega)$, we first have

$$\|F(u_n)-F(u)\|_{L^p(\Omega)}\le \|F'\|_{L^\infty(\mathbb{R})}\|u_n-u\|_{L^p(\Omega)}$$

which gives $F(u_n)\to F(u)$ in $L^p(\Omega)$. By boundedness of F', the convergence

$$F'(u_n)Du\to F'(u)Du \quad \text{in } L^p(\Omega;\mathbb{R}^n)$$

can be proved by dominated convergence. Finally, one has

$$\begin{aligned}\lim_n \|F'(u_n)Du_n-F'(u)Du\|_{L^p(\Omega;\mathbb{R}^n)} &= \lim_n \|F'(u_n)Du_n-F'(u_n)Du\|_{L^p(\Omega;\mathbb{R}^n)}\\ &\le \|F'\|_{L^\infty(\mathbb{R})}\lim_n \|Du_n-Du\|_{L^p(\Omega;\mathbb{R}^n)}=0\,,\end{aligned}$$

whence it follows that $F(u)\in W^{1,p}(\Omega)$, and that the weak gradient is given by $F'(u)Du$. □

Exercise 3. Let Ω be a bounded open subset of $\mathbb{R}^n$, and $u\in W^{1,p}(\Omega)$. Define as usual $u^+(x):=\max\{u(x),0\}$, and $u^-(x):=\max\{-u(x),0\}$. Show that u^+, u^-, and $|u|\in W^{1,p}(\Omega)$, too, and compute the corresponding weak gradients.

Hint: $u^+=\lim_{\varepsilon\to0}F_\varepsilon(u)$, with

$$F_\varepsilon(z):=\begin{cases}(z^2+\varepsilon^2)^{\frac12}-\varepsilon & \text{if} \quad z\ge0\\ 0 \quad \text{if} \quad z<0\,.\end{cases}$$

Solution: Take F_ε as in the hint, and observe that $F_\varepsilon\in C^1\cap W^{1,\infty}(\mathbb{R})$. Therefore, the previous exercise gives that $F_\varepsilon(u)\in W^{1,p}(\Omega)$ and that

$$D(F_\varepsilon(u))=F'_\varepsilon(u)Du=u^+(\varepsilon^2+u^2)^{-\frac12}Du\,,$$

where the last equality follows by a direct computation. Since $u^+(\varepsilon^2+u^2)^{-\frac12}\to\chi_{\{u>0\}}$ a.e. in Ω, by dominated convergence we get that

$$D(F_\varepsilon(u))\to\chi_{\{u>0\}}Du$$

in $L^p(\Omega;\mathbb{R}^n)$. It follows that u^+ belongs to $W^{1,p}(\Omega)$, and that

$$D(u^+)=\chi_{\{u>0\}}Du\,.$$

Since $u^-=u^+-u$ and $|u|=u^+-u^-$ we get that u^-, and $|u|\in W^{1,p}(\Omega)$, too, and the weak gradients are given by

$$D(u^-)=-\chi_{\{u\le0\}}Du \quad \text{and} \quad D(|u|)=\chi_{\{u>0\}}Du-\chi_{\{u\le0\}}Du\,.$$

□

8. Exercises PDE 9-11.01.13

Exercise 1. Prove the following multiplicative form of the trace inequality: if $u\in W^{1,2}(\mathbb{R}^n_+)$, denoting with Tu the trace of u on the hyperplane $\mathbb{R}^{N-1}$ one has

$$\|Tu\|^2_{L^2(\mathbb{R}^{N-1})}\le C\|u\|_{L^2(\mathbb{R}^N_+)}\|Du\|_{L^2(\mathbb{R}^N_+)}\,.$$

Solution: Being $\mathbb{R}^n_+$ an extension domain, for every $u\in W^{1,2}(\mathbb{R}^n_+)$ there exists a sequence u_n of functions in $C^\infty_c(\mathbb{R}^n)$ converging to u in $W^{1,2}(\mathbb{R}^n_+)$. Now, for every $\hat{x}\in\mathbb{R}^{N-1}$ by the fundamental Theorem of calculus one has

$$u_n^2(\hat{x},0)=-\int_0^{+\infty}\frac{\partial}{\partial x_N}u_n^2(\hat{x},x_N)\,dx_N=-2\int_0^{+\infty}u_n(\hat{x},x_N)\frac{\partial}{\partial x_N}u_n(\hat{x},x_N)\,dx_N$$

for every $n\in\mathbb{N}$. This implies by Fubini's Theorem

$$\int_{\mathbb{R}^{N-1}}u_n^2(\hat{x},0)\,d\hat{x}=-2\int_{\mathbb{R}^n_+}u_n(x)\frac{\partial}{\partial x_N}u_n(x)\,dx\,.$$

Since for every n and every $\hat{x}$ the trace $Tu_n(\hat{x})$ of u_n on the hyperplane $\mathbb{R}^{N-1}$ is simply equal to $u_n(\hat{x},0)$, the previous equality and the Cauchy-Schwarz inequality yield

$$\|Tu_n\|^2_{L^2(\mathbb{R}^{N-1})} \leq 2\,\|u_n\|_{L^2(\mathbb{R}^n_+)}\,\|Du_n\|_{L^2(\mathbb{R}^n_+)}\,.$$

Since u_n is converging to u in $W^{1,2}(\mathbb{R}^n_+)$, by continuity of the trace operator Tu_n is converging to Tu in $L^2(\mathbb{R}^{N-1})$, so that we get

$$\|Tu\|^2_{L^2(\mathbb{R}^{N-1})} \leq 2\,\|u\|_{L^2(\mathbb{R}^n_+)}\,\|Du\|_{L^2(\mathbb{R}^n_+)}\,.$$

by letting n going to $+\infty$, as required. □

Exercise 2. Let Ω be an open set of $\mathbb{R}^n$, and Γ a C^1 hypersurface such that $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ with Ω_1 and Ω_2 open connected disjoint.

(a) Let $f_1 \in C^1(\overline{\Omega}_1)$ and $f_2 \in C^1(\overline{\Omega}_2)$. Under which conditions the function $f \in L^\infty(\Omega)$ defined by $f = f_1$ in Ω_1 and $f = f_2$ in Ω_2 belongs to $W^{1,\infty}(\Omega)$?

(b) Let $p \geq 1$, $f_1 \in W^{1,p}(\Omega_1)$ and $f_2 \in W^{1,p}(\Omega_2)$. Under which conditions the function $f \in L^p(\Omega)$ defined by $f = f_1$ in Ω_1 and $f = f_2$ in Ω_2 belongs to $W^{1,p}(\Omega)$?

Solution: (a) Denote with ν_Γ the normal vector to the hypersurface Γ pointing from Ω_1 to Ω_2. By the Divergence Theorem we have, for every $\varphi \in C^\infty_c(\Omega)$,

$$\int_\Omega f(x)\mathrm{div}\,\varphi(x)\,dx = \int_{\Omega_1} f_1(x)\mathrm{div}\,\varphi(x)\,dx + \int_{\Omega_2} f_2(x)\mathrm{div}\,\varphi(x)\,dx =$$
$$-\int_{\Omega_1} Df_1(x)\cdot\varphi(x)\,dx - \int_{\Omega_2} Df_2(x)\cdot\varphi(x)\,dx + \int_\Gamma (f_1(\xi) - f_2(\xi))\,\varphi(\xi)\cdot\nu_\Gamma(\xi)\,dS(\xi)$$

therefore $f \in W^{1,\infty}(\Omega)$ if and only if $f_1 = f_2$ on Γ. In that case one also has

$$Df = \begin{cases} Df_1 \text{ in } \Omega_1 \\ Df_2 \text{ in } \Omega_2\,. \end{cases}$$

(b) It suffices to repeat the previous reasonings with suitable using of the notion of trace to deduce that the required condition is $Tf_1 = Tf_2$ a.e. in Γ, where T denotes the trace operator from $W^{1,p}(\Omega_1)$ to $L^p(\partial\Omega_1)$, and from $W^{1,p}(\Omega_2)$ to $L^p(\partial\Omega_2)$, respectively. □

Exercise 3. Give an example of a connected open set $\Omega \in \mathbb{R}^n$ and of a function $u \in W^{1,\infty}(\Omega)$ such that u is not Lipschitz continuous on Ω.

Solution: It simply suffices to modify the example in Exercise 1, 19.12.12. Namely, set $\Omega := B(0,2) \setminus \{x_N = 0,\, |x| \leq \frac{1}{2}\}$, which is connected, and let for instance

$$f(x) := \begin{cases} (|x|^2 - \frac{1}{4})^2 & \text{if } x_N > 0,\ |x| \leq \frac{1}{2} \\ 0 \quad \text{elsewhere}\,. \end{cases}$$

Denoting with $B\big(0,\frac{1}{2}\big)^+$ the upper hemisphere $B\big(0,\frac{1}{2}\big) \cap \{x_N > 0\}$, one has by the Divergence theorem that for any $\varphi \in C^\infty_c(\Omega;\mathbb{R}^n)$

$$\int_\Omega f(x)\mathrm{div}\,\varphi(x)\,dx = \int_{B\left(0,\frac{1}{2}\right)^+} f(x)\mathrm{div}\,\varphi(x)\,dx = -4\int_{B\left(0,\frac{1}{2}\right)^+} (|x|^2 - \tfrac{1}{4})\,x\cdot\varphi(x)\,dx\,;$$

indeed, no boundary term appears since either f or φ are 0 on $\partial B\big(0,\frac{1}{2}\big)^+$. It follows that $f \in W^{1,\infty}(\Omega)$. On the other hand, for every $0 < \varepsilon < \frac{1}{2}$, one has

$$\frac{|f(0,\ldots,0,\varepsilon) - f(0,\ldots,0,-\varepsilon)|}{|(0,\ldots,0,\varepsilon) - (0,\ldots,0,-\varepsilon)|} = \frac{(\varepsilon^2 - \frac{1}{4})^2}{2\varepsilon}$$

which is unbounded when ε is close to 0. Therefore f cannot be Lipschitz continuous. □

9. Exercises PDE 16-18.01.13

Exercise 1. Let Ω be a bounded domain with smooth boundary and $u \in W^{1,1}(\Omega)$. Assume that $Du \in L^p(\Omega)$ for some $1 < p < \infty$. Prove that $u \in W^{1,p}(\Omega)$.

Hint: Prove initially that $u \in L^p_{loc}(\Omega)$, to understand the exercise. Taking a closer look to the proof of Theorem 3, Section 5.3.3 of Evans' book can help near to the boundary.

Solution*: First step:* let us prove that $u \in L^p_{loc}(\Omega)$, which requires no regularity of the boundary. Define $u_n = \varrho_n \star u \in C^\infty(\mathbb{R}^n)$, and fix a smooth open subset $\Omega' \subset\subset \Omega$. For n large, depending on Ω', it holds $Du_n = \varrho_n \star Du$ in Ω' (the formula $Du_n = D\varrho_n \star u$ is instead true on the whole $\mathbb{R}^n$: why this difference?). We therefore deduce by Jensen's inequality (how?) that

$$\|Du_n\|_{L^p(\Omega')} \leq \|Du\|_{L^p(\Omega)} . \tag{9.1}$$

By Poincaré-Wirtinger and (9.1), if $\bar{u}_{n,\Omega'}$ denotes the integral mean of u_n on Ω' we get that there exists a constant $C_{\Omega'}$ depending on Ω' such that

$$\int_{\Omega'} |u_n(x) - \bar{u}_{n,\Omega'}|^p \, dx \leq C_{\Omega'} \|Du\|^p_{L^p(\Omega)} .$$

Since u_n is converging to u in $L^1(\Omega')$, by Fatou's Lemma, denoting with $\bar{u}_{\Omega'}$ the integral mean of u on Ω', we arrive at

$$\int_{\Omega'} |u(x) - \bar{u}_{\Omega'}|^p \, dx \leq C_{\Omega'} \|Du\|^p_{L^p(\Omega)}$$

and this implies the claim.

Second step: suitably changing coordinates like in Theorem 3, Section 5.3.3 of Evans' book, for every $x_0 \in \partial\Omega$ one can find a neighborhood $V := B(x_0, r) \cap \Omega$ and a positive number $\lambda > 0$ such that the ball $B(x + \frac{\lambda}{n} e_N, \frac{1}{n}) \subset \Omega$ for all $x \in V$. Therefore, if one defines $u_n(x) = u(x + \frac{\lambda}{n} e_N)$ one has $u_n \in W^{1,1}(V)$ and since we only did a translation of the argument

$$\|Du_n\|_{L^p(V)} \leq \|Du\|_{L^p(\Omega)} . \tag{9.2}$$

Now, defining $v_n = \varrho_n \star u_n \in C^\infty(\mathbb{R}^n)$, by condition $B(x + \frac{\lambda}{n} e_N, \frac{1}{n}) \subset \Omega$ it holds $Dv_n = \varrho_n \star Du_n$. Notice also that u_n and thus v_n converge to u in $L^1(V)$. Therefore, the same argument used in the first step plus (9.2) proves that u in $L^p(V)$.

Third step: Ω can be written as a finite union $\Omega = \cup_{i=1}^m V_i$ where $V_0 \subset\subset \Omega$ and each V_i, $i \geq 1$ is such that the second step can be applied. Since

$$\int_\Omega |u(x)|^p \, dx \leq \sum_{i=1}^m \int_{V_i} |u(x)|^p \, dx$$

the finiteness of the right-hand side gives $u \in L^p(\Omega)$. Since $Du \in L^p(\Omega)$ by the hypothesis, the exercise is concluded. □

Exercise 2. *The geometric counterpart of the Sobolev inequality.*

(a) (optional, if not simply assume the result). Let $f \in L^1(\mathbb{R}^n; \mathbb{R}^m)$. Prove that

$$\int_{\mathbb{R}^n} |f(x)| \, dx = \sup \Big\{ \int_{\mathbb{R}^n} f(x) \cdot \varphi(x) \, dx : \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^m), \|\varphi\|_\infty \leq 1 \Big\} . \tag{9.3}$$

(b) Let $C_{1,N}$ the Sobolev constant in $\mathbb{R}^n$, i.e. the least possible constant such that the Sobolev inequality

$$\|u\|_{\frac{N}{N-1}} \leq C \|Du\|_1$$

holds for every $u \in W^{1,1}(\mathbb{R}^n)$. Prove the following isoperimetric inequality: for every smooth bounded open set $\Omega \subset \mathbb{R}^n$ one has

$$|\Omega|^{\frac{N-1}{N}} \leq C_{1,N} \mathcal{H}^{n-1}(\partial\Omega) , \tag{9.4}$$

where $|\Omega|$ is the Lebesgue measure of Ω and $\mathcal{H}^{n-1}(\partial\Omega)$ is the (Hausdorff, or surface) measure of the boundary. *Hint*: approximate the characteristic function χ_Ω by convolution and use (9.3) and the divergence theorem to estimate the L^1 norm of the gradients.

Solution: (a) Define

$$f_1(x) := \begin{cases} \frac{f(x)}{|f(x)|} & \text{if} \quad f(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let ϱ_n be the usual mollifiers, and, given a sequence φ_n of $C_c^\infty(\mathbb{R}^n)$ functions with $0 \le \varphi_n \le 1$ with $\varphi_n(x) \to 1$ for every x, set $g_n := \varphi_n(\varrho_n \star f_1)$. It is clear that $g_n \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^m)$ and that $\|g_n\|_\infty \le 1$. Furthermore g_n converges to f_1 a.e. Observing that $f(x) \cdot f_1(x) = |f(x)|$ for every x, by dominated convergence one has

$$\int_{\mathbb{R}^n} |f(x)|\, dx = \lim_n \int_{\mathbb{R}^n} f(x) \cdot g_n(x)\, dx \le \sup\Big\{ \int_{\mathbb{R}^n} f(x) \cdot \varphi(x)\, dx : \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^m), \|\varphi\|_\infty \le 1 \Big\}.$$

The converse inequality is trivial so that we get (9.3).

(b) By (9.3) and integrating by parts we get that for every $u \in W^{1,1}(\mathbb{R}^n)$ one has

$$\|Du\|_1 = \sup\Big\{ \int_{\mathbb{R}^n} u \operatorname{div} \varphi : \varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n),\ \|\varphi\|_\infty \le 1 \Big\}. \tag{9.5}$$

Now, if ϱ_n are the usual mollifiers, we have for every $\varphi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ with $\|\varphi\|_\infty \le 1$ that also $\|\varrho_n \star \varphi\|_\infty \le 1$. Therefore, by symmetry of the mollifiers, standard properties of the convolution and also using the divergence theorem, we have

$$\int_{\mathbb{R}^n} (\varrho_n \star \chi_\Omega)(x) \operatorname{div} \varphi(x)\, dx = \int_{\mathbb{R}^n} \chi_\Omega(x) \operatorname{div}(\varrho_n \star \varphi)(x)\, dx = \int_\Omega \operatorname{div}(\varrho_n \star \varphi)(x)\, dx =$$
$$\int_{\partial\Omega} (\varrho_n \star \varphi)(\xi) \cdot \nu(\xi)\, d\mathcal{H}^{n-1}(\xi) \le \mathcal{H}^{n-1}(\partial\Omega)$$

for every smooth bounded open set $\Omega \subset \mathbb{R}^n$. It follows from this and (9.5) that for every $n \in \mathbb{N}$ one has

$$\|D(\varrho_n \star \chi_\Omega)\|_1 \le \mathcal{H}^{n-1}(\partial\Omega) \tag{9.6}$$

for every smooth bounded open set $\Omega \subset \mathbb{R}^n$.

Now, combining (9.6) with the Sobolev inequality gives

$$\|\varrho_n \star \chi_\Omega\|_{\frac{N}{N-1}} \le C_{1,N} \mathcal{H}^{n-1}(\partial\Omega)$$

for every smooth bounded open set $\Omega \subset \mathbb{R}^n$ and every $n \in \mathbb{N}$. Letting n to $+\infty$ we get

$$\|\chi_\Omega\|_{\frac{N}{N-1}} \le C_{1,N} \mathcal{H}^{n-1}(\partial\Omega)$$

which is exactly (9.4). □

Exercise 3. Let $p > 2$. Show with a counterexample that the Morrey's Imbedding Theorem is not verified in the nonsmooth two-dimensional domains

$$D_\alpha := \{(x,y) : 0 < x < 1, \quad 0 < y < x^\alpha\}$$

for α sufficiently large.

Hint: prove that there are unbounded funtions in $W^{1,p}(D_\alpha)$.

Solution: Let $\alpha > p - 1$. We can choose $\gamma > 0$ such that $\frac{\alpha+1}{\gamma+1} > p$, which is equivalent to

$$\alpha - (\gamma + 1)p > -1\,. \tag{9.7}$$

Now, consider the function

$$u_\gamma(x,y) := x^{-\gamma}\,.$$

The function u_γ is unbounded in D_α so that it cannot belong to a space of Hölder functions. On the other hand by Fubini's theorem one has

$$\int_{D_\alpha} |u_\gamma(x,y)|^p\, dxdy = \int_0^1 x^{\alpha - \gamma p}\, dx < +\infty$$

since by (9.7) one gets $\alpha - \gamma p > -1$. Similarly, being $Du(x,y) = (-\gamma x^{-(1+\gamma)}, 0)$ one gets

$$\int_{D_\alpha} |Du_\gamma(x,y)|^p \, dxdy = \gamma^p \int_0^1 x^{\alpha-(\gamma+1)p} \, dx < +\infty$$

again using Fubini's theorem and (9.7). Therefore $u_\gamma \in W^{1,p}(D_\alpha)$ but it cannot satisfy Morrey's imbedding Theorem.

□

10. Exercises PDE 23-25.01.13

Exercise 1. *Fourier analysis in Sobolev spaces.* Let $f \in L^2((0,\pi))$. Define for every $N \in \mathbb{N}$

$$f_N(t) := \frac{2}{\pi} \sum_{n=1}^{N} a_n \sin(nt)$$

where $a_n = \int_0^\pi f(t) \sin(nt)\, dt$. It is known that f_N converges to f in $L^2((0,\pi))$ (if you need to convince yourself that no cosine is needed to have L^2 convergence, simply apply the general theorem to the odd extension of f to $[-\pi,\pi]$).

(a) Assume that $f \in H_0^1((0,\pi))$. Show that f_N converges to f uniformly in $[0,\pi]$. *Hint*: prove f_N to be a Cauchy sequence in some Sobolev space.

(b) If $f \in H_0^1((0,\pi)) \cap H^2((0,\pi))$ prove that there exists a constant C such that

$$\|f_N - f\|_{H^1} \leq \frac{C}{N} \|f\|_{H^2} \,. \tag{10.1}$$

Solution*:* (a) Since $f(0) = f(\pi) = 0$ integrating by parts we obtain

$$na_n = -\int_0^\pi f'(t)\, \cos(nt)\, dt \tag{10.2}$$

for every $n \in \mathbb{N}$. Since $\{\sqrt{\frac{2}{\pi}} \cos(nt) : n \in \mathbb{N}\}$ is an orthonormal system of $L^2(0,\pi)$, Bessel's inequality gives

$$\sum_{n=1}^{N} n^2 a_n^2 = \sum_{n=1}^{N} \Big(\int_0^\pi f'(t)\, \cos(nt)\, dt \Big)^2 \leq \frac{\pi}{2} \|f'\|_2^2$$

for every $N \in \mathbb{N}$. Being a convergent series, $\sum_{n=1}^{N} n^2 a_n^2$ is a Cauchy sequence.

On the other hand, since $\{\sqrt{\frac{2}{\pi}} \cos(nt) : n \in \mathbb{N}\}$ is an orthonormal system of $L^2(0,\pi)$, for every $N \in \mathbb{N}$ and every $M \in \mathbb{N}$ with $M \geq N$, one has

$$\|f'_M - f'_N\|_2^2 = \| \sum_{n=N}^{M} na_n \cos(n\cdot) \|_2^2 = \sum_{n=N}^{M} n^2 a_n^2 \| \cos(n\cdot)\|_2^2 = \sum_{n=N}^{M} n^2 a_n^2 \,. \tag{10.3}$$

It follows that f_N is a Cauchy sequence in $L^2((0,\pi))$. Since f_N was converging to f in $L^2((0,\pi))$, combining the two gives that f_N is converging to f in $H_0^1((0,\pi))$, and therefore uniformly by Morrey's imbedding Theorem.

(b) If $f \in H^2((0,\pi))$ we can once more integrate by parts in (10.2), obtaining

$$\int_0^\pi f'(t)\, \cos(nt)\, dt = -\frac{1}{n} \int_0^\pi f''(t)\, \sin(nt)\, dt$$

so that

$$n^2 a_n = \int_0^\pi f''(t)\, \sin(nt)\, dt$$

and, again using Bessel's inequality,

$$\sum_{n=1}^{N} n^4 a_n^2 \leq \frac{\pi}{2} \|f''\|_2^2$$

for every $N \in \mathbb{N}$. Combining this with (10.3) for every $N \in \mathbb{N}$ and every $M \in \mathbb{N}$ with $M \geq N$, one has

$$\|f'_M - f'_N\|_2^2 = \sum_{n=N}^{M} n^2 a_n^2 \leq \frac{1}{N^2} \sum_{n=N}^{M} n^4 a_n^2 \leq \frac{\pi}{2N^2} \|f''\|_2^2 .$$

When M goes to $+\infty$ we therefore obtain

$$\|f' - f'_N\|_2 \leq \sqrt{\frac{\pi}{2}} \frac{1}{N} \|f''\|_2$$

Since by Poincaré's inequality

$$\|f - f_N\|_2 \leq \pi \|f' - f'_N\|_2$$

we get that

$$\|f - f_N\|_{H_0^1((0,\pi))} \leq \sqrt{\frac{\pi}{2}} (1+\pi) \frac{1}{N} \|f''\|_2$$

as required. □

Exercise 2. *Céa's Lemma:* Consider a symmetric, bounded, coercive bilinear form $B \colon H \times H \to \mathbb{R}$ on a Hilbert space H, that is

$$B(u,v) = B(v,u), \quad B(u,v) \leq C_1 \|u\|_H \|v\|_H , \quad B(u,u) \geq c_0 \|u\|_H^2 .$$

(a) Fix a linear continuous functional $F \colon H \to \mathbb{R}$. Justify that there exists a unique $u \in H$ such that

$$B(u,v) = F(v)$$

for every $v \in H$.

(b) Fix additionally an N-dimensional subspace H_N of H. Justify the following fact: there exist a unique approximate solution $u_N \in H_N$, that is a unique $u_N \in H_N$ such that

$$B(u_N, v_N) = F(v_N)$$

for every $v_N \in H_N$.

(c) Prove the following equality

$$B(u - u_N, u - u_N) + B(v_N - u_N, v_N - u_N) = B(u - v_N, u - v_N) \tag{10.4}$$

for every $v_N \in H_N$. Deduce Céa's estimate:

$$\|u - u_N\|_H \leq \sqrt{\tfrac{C_1}{c_0}} \inf\{\|u - v_N\| : v_N \in H_N\} . \tag{10.5}$$

(d) Consider now $H = H_0^1((0,\pi))$ and $H_N := \operatorname{span}\{t \mapsto \sin(nt) : n = 1, \dots, N\}$. For $a(x) \in C^1[0,\pi]$ with $1 \leq a(x) \leq 2$ for every x and $f \in L^2((0,\pi))$ set

$$B(u,v) := \int_0^\pi a(t)\, u'(t)\, v'(t)\, dt \tag{10.6}$$

and $F(v) := \int_0^\pi f(t)\, v(t)\, dt$. Check that u and u_N as in (a) and (b), respectively, exist, and show that $u_N \to u$ in $H_0^1((0,\pi))$ as N goes to $+\infty$. If additionally $u \in H^2((0,\pi))$ show that there exists a constant C such that

$$\|u - u_N\|_{H^1} \leq \frac{C}{N} \|u\|_{H^2} .$$

Solution*:* (a) follows from the Lax-Milgram theorem. The same holds for (b) since the restriction of B to the Hilbert subspace $H_N \times H_N$ is clearly bilinear, coercive and continuous.

(c) By bilinearity and simmetry of the form B we have

$$\begin{aligned} B(u - u_N, u - u_N) &= B(u,u) - 2B(u,u_N) + B(u_N,u_N) \\ B(v_N - u_N, v_N - u_N) &= B(v_N,v_N) - 2B(u_N,v_N) + B(u_N,u_N) \\ B(u - v_N, u - v_N) &= B(u,u) - 2B(u,v_N) + B(v_N,v_N) \end{aligned}$$

therefore (10.4) is equivalent to prove that

$$2(B(u_N, u_N) - B(u, u_N) + B(u, v_N) - B(u_N, v_N)) = 0\,.$$

This can be easily proved to be true by observing that by definition of u and u_N, for every $v_N \in H_N$ one has

$$B(u, v_N) = F(v_N) = B(u_N, v_N)$$

and in particular, for $v_N = u_N$,

$$B(u, u_N) = B(u_N, u_N)\,.$$

Now, by (10.4), and the coercivity and continuity of B we get for every $v_N \in H_N$

$$c_0\|u - u_N\|_H^2 \le B(u - u_N, u - u_N) \le B(u - u_N, u - u_N) + B(v_N - u_N, v_N - u_N) = B(u - v_N, u - v_N) \le C_1\|u - v_N\|_H^2$$

whence

$$\|u - u_N\|_H \le \sqrt{\tfrac{C_1}{c_0}}\|u - v_N\|_H$$

for every $v_N \in H_N$. Taking the infimum in the rigth-hand side gives (10.5).

(d) To prove existence of u and u_N when B is given by (10.6), it suffices to check that the Lax-Milgram theorem can be applied. The only relevant point to this end is to prove coercivity of B, which follows from the Poincaré inequality. Namely, one has

$$B(u, u) \ge \|u'\|_2^2 \ge \frac{1}{1+\pi^2}\|u\|_{H^1}^2\,.$$

It follows now from the previous exercise that

$$\inf\{\|u - v_N\|_{H_0^1((0,\pi))} : v_N \in H_N\} \to 0$$

as N goes to $+\infty$. In particular, by (10.1), if $u \in H^2$ one has for every $N \in \mathbb{N}$

$$\inf\{\|u - v_N\|_{H_0^1((0,\pi))} : v_N \in H_N\} \le \frac{C}{N}\|u\|_{H^2}\,.$$

Combining these two facts with (10.5) gives the conclusion. □

Exercise 3. Consider a bounded smooth open subset $\Omega \subset \mathbb{R}^n$, and $\Gamma_D \subset \partial\Omega$ with $\mathcal{H}^{n-1}(\Gamma_D) > 0$. Let

$$H^1_{\Gamma_D}(\Omega) := \{u \in H^1(\Omega), Tu = 0 \text{ on } \Gamma_D\}\,,$$

with T the trace operator. For $\lambda > 0$ define

$$B_\lambda(u, v) = \int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx - \frac{1}{\lambda}\int_{\partial\Omega} Tu(\xi)\,Tv(\xi)\,d\mathcal{H}^{n-1}(\xi)$$

for u and $v \in H^1_{\Gamma_D}(\Omega)$. Prove that, when λ is sufficiently large, for every $f \in L^2(\Omega)$ there exists a unique $u_f \in H^1_{\Gamma_D}(\Omega)$ such that

$$B_\lambda(u_f, v) = \int_\Omega f(x)\,v(x)\,dx$$

for every $v \in H^1_{\Gamma_D}(\Omega)$. *Hint:* does the Poincaré inequality apply in $H^1_{\Gamma_D}(\Omega)$?

Solution*:* Following the hint, let us first prove that there exists a constant C such that

$$\|u\|_{L^2(\Omega)} \le C\|\nabla u\|_{L^2(\Omega)} \tag{10.7}$$

for every $u \in H^1_{\Gamma_D}(\Omega)$. To prove this, it suffices to argue as in the proof of the Poincaré-Wirtinger inequality (Theorem 1, Section 5.8.1 in Evans'book): contradicting (10.7) would produce the existence of a sequence u_n in $H^1_{\Gamma_D}(\Omega)$ with $\|u_n\|\|_{L^2(\Omega)} = 1$ for every n which is converging to a constant c in $H^1(\Omega)$. By continuity of the trace operator, $H^1_{\Gamma_D}(\Omega)$ is a Hilbert subspace of $H^1(\Omega)$, so it must be $c = 0$, since this one is the only constant in $H^1_{\Gamma_D}(\Omega)$. On the other hand, by strong L^2 convergence we must have $\|c\|_{L^2(\Omega)} = c|\Omega|^{\frac{1}{2}} = 1$, and this gives a contradiction.

Now, let C be given by (10.7), and let C_T the continuity constant of the trace operator. Assume that $\lambda > C_T(1+C^2)$. Then for every $u \in H^1_{\Gamma_D}(\Omega)$ one has, by the continuity of the trace and by (10.7), that

$$B_\lambda(u,u) = \|\nabla u\|^2_{L^2(\Omega)} - \frac{1}{\lambda}\|Tu\|^2_{L^2(\partial\Omega)} \geq \frac{1}{1+C^2}\|u\|^2_{H^1(\Omega)} - \frac{C_T}{\lambda}\|u\|^2_{H^1(\Omega)} = c_0\|u\|^2_{H^1(\Omega)}$$

with $c_0 := \frac{1}{1+C^2} - \frac{C_T}{\lambda} > 0$ due to the assumption on λ. Therefore for $\lambda > C_T(1+C^2)$ the bilinear form B_λ is coercive on $H^1_{\Gamma_D}(\Omega)$. For any $\lambda > 0$ one has also by the Cauchy-Schwarz inequality and the continuity of the trace operator that

$$B_\lambda(u,v) \leq (1 + \frac{C_T}{\lambda})\|u\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}$$

so that B_λ is continuous. The conclusion follows by the Lax-Milgram theorem. □

11. Exercises PDE 30.01.13-01.02.13

Exercise 1. Let H be an Hilbert space, $a : H \times H \to \mathbb{R}$ a bilinear, symmetric, coercive, and continuous form, and $L : H \to \mathbb{R}$ a linear continuous functional. Prove that $u \in H$ solves

$$a(u,v) = L(v) \quad \text{for every } v \in H \tag{11.1}$$

if and only if

$$\frac{1}{2}a(u,u) - L(u) = \min\left\{\frac{1}{2}a(v,v) - L(v) : v \in H\right\}. \tag{11.2}$$

Hint: if u is a minimizer, fix $v \in H$ and consider the function $F(t) := \frac{1}{2}a(u+tv, u+ tv) - L(u+tv)$, $t \in \mathbb{R}$.

Solution*: "If" part:* assume that $u \in H$ satisfies (11.2). Fix $v \in H$ and consider the function $F(t) := \frac{1}{2}a(u+tv, u+ tv) - L(u+tv)$, $t \in \mathbb{R}$. The function F attains then its minimum value for $t = 0$. Since by a direct computation

$$F(t) = \frac{1}{2}a(u,u) - L(u) + t(a(u,v) - L(v)) + \frac{t^2}{2}a(v,v) \tag{11.3}$$

imposing $F'(0) = 0$ we get (11.1).

"Only if" part: assume that $u \in H$ satisfies (11.1). Fix $v \in H$ and, again, consider the function $F(t) := \frac{1}{2}a(u+tv, u+ tv) - L(u+tv)$, $t \in \mathbb{R}$. By (11.3) and (11.1) we obtain

$$F(t) = \frac{1}{2}a(u,u) - L(u) + \frac{t^2}{2}a(v,v).$$

By coerciveness of a, $a(v,v) \geq 0$ which implies that F attains its minimum at 0. In particular $F(0) \leq F(1)$, therefore

$$\frac{1}{2}a(u,u) - L(u) \leq \frac{1}{2}a(u+v, u+v) - L(u+v) \tag{11.4}$$

for every $v \in H$. Now, for an arbitrary $w \in H$, applying (11.4) with $v = w - u$ one gets

$$\frac{1}{2}a(u,u) - L(u) \leq \frac{1}{2}a(w,w) - L(w)$$

that is (11.2). □

Exercise 2. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with smooth boundary, and $f \in L^2(\Omega)$. We say that $u \in H^1(\Omega)$ is a weak solution of the Neumann problem

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \end{cases} \tag{11.5}$$

if

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega f(x) v(x)\, dx \tag{11.6}$$

for any $v \in H^1(\Omega)$.

(a) Justify that actually (11.6) is a weak formulation of the problem (11.5) by showing that if u solves (11.6) and in addition $u \in H^2(\Omega)$ then $-\Delta u = f$ in $L^2(\Omega)$ and $\frac{\partial u}{\partial \nu} = 0$ in $L^2(\partial\Omega)$ in the sense of traces.

(b) Show that a necessary and sufficient condition for the existence of a solution of (11.6) is

$$\int_\Omega f(x)\, dx = 0\,. \tag{11.7}$$

Hint: use Lax-Milgram's theorem in the Hilbert space

$$H := \{u \in H^1(\Omega) : \int_\Omega u(x)\, dx = 0\}\,. \tag{11.8}$$

(c) Let $g \in L^2(\partial\Omega)$. Generalise part (a) and (b) to the nonhomogeneuous Neumann problem

$$\begin{cases} -\Delta u = f \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = g \text{ on } \partial\Omega \end{cases} \tag{11.9}$$

by finding its weak formulation and a necessary and sufficient condition for existence of a weak solution.

Solution*:* (a) If u solves (11.6) and in addition $u \in H^2(\Omega)$, for every $v \in C_c^\infty(\Omega)$ integrating by parts and using (11.6) we have

$$-\int_\Omega \Delta u(x) v(x)\, dx = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega f(x) v(x)\, dx$$

which implies $-\Delta u = f$ in $L^2(\Omega)$. Using this additional information, integrating by parts and using again (11.6) we get for every $v \in H^1(\Omega)$

$$\int_\Omega f(x) v(x)\, dx = \\ -\int_\Omega \Delta u(x) v(x)\, dx = \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(\xi) v(\xi)\, d\mathcal{H}^{n-1}(\xi) = \\ \int_\Omega f(x) v(x)\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(\xi) v(\xi)\, d\mathcal{H}^{n-1}(\xi)\,,$$

which gives $\frac{\partial u}{\partial \nu} = 0$ in $L^2(\partial\Omega)$.

Observe in addition that via integration by parts also the converse holds: if $u \in H^2(\Omega)$ solves (11.5), then (11.6) holds.

(b) Taking $v = 1$ in (11.6) gives that (11.7) is necessary. To prove that it is sufficient, defining the subspace H as in (11.8), we first prove that, if $u \in H$ satisfies

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega f(x) v(x)\, dx \tag{11.10}$$

for every $v \in H$, then (11.6) holds. Indeed, when $v \in H^1(\Omega)$, then $\tilde{v} = v - \frac{1}{|\Omega|}\int_\Omega v(x)\, dx$ belongs to H; furthermore, $\nabla \tilde{v} = \nabla v$ and, by (11.7),

$$\int_\Omega f(x) v(x)\, dx = \int_\Omega f(x) \tilde{v}(x)\, dx\,.$$

Therefore (11.10) implies (11.6). Now, existence of a unique solution to (11.10) in H follows by the Lax-Milgram theorem once it is checked that the bilinear form $B\colon H\times H\to\mathbb{R}$ defined by

$$B(u,v):=\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx$$

is continuous and coercive. Continuity is obvious, while coercivity follows by the Poincaré-Wirtinger inequality

$$\|\nabla u\|_2\geq C_\Omega\|u\|_2$$

for every $u\in H$ (add the details!).

(c) The solution is very similar to the previous case, so I only sketch it. Integrating by parts, one easily gets that a smooth solution of (11.9) satisfies

$$\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx=\int_\Omega f(x)v(x)\,dx+\int_{\partial\Omega} g(\xi)v(\xi)\,d\mathcal{H}^{n-1}(\xi)\,. \tag{11.11}$$

This is therefore our candidate weak formulation. Testing (11.11) on C_c^∞ functions we get that if u solves it and in addition $u\in H^2(\Omega)$ then $-\Delta u=f$ in $L^2(\Omega)$. Using this additional information, integrating by parts and using again (11.11) we get for every $v\in H^1(\Omega)$

$$\begin{gathered}\int_\Omega f(x)v(x)\,dx=\\ -\int_\Omega \Delta u(x)v(x)\,dx=\int_\Omega\nabla u(x)\cdot\nabla v(x)\,dx-\int_{\partial\Omega}\frac{\partial u}{\partial\nu}(\xi)v(\xi)\,d\mathcal{H}^{n-1}(\xi)=\\ \int_\Omega f(x)v(x)\,dx-\int_{\partial\Omega}\Big(\frac{\partial u}{\partial\nu}(\xi)-g(\xi)\Big)v(\xi)\,d\mathcal{H}^{n-1}(\xi)\,,\end{gathered}$$

which gives $\frac{\partial u}{\partial\nu}=g$ in $L^2(\partial\Omega)$.

We then show that (11.11) has a solution if and only if

$$\int_\Omega f(x)\,dx+\int_{\partial\Omega} g(\xi)\,d\mathcal{H}^{n-1}(\xi)=0\,. \tag{11.12}$$

Indeed, taking $v=1$ in (11.11) produces (11.12) as a necessary condition. On the other hand, if (11.12) holds, exactly as in part (b) we can show that, defining the subspace H as in (11.8), if $u\in H$ satisfies

$$\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx=\int_\Omega f(x)v(x)\,dx+\int_{\partial\Omega} g(\xi)v(\xi)\,d\mathcal{H}^{n-1}(\xi)\,. \tag{11.13}$$

for every $v\in H$, then (11.11) holds. Existence of a solution to (11.13) in H follows again by the Lax-Milgram Theorem; indeed, the bilinear form $B\colon H\times H\to\mathbb{R}$ defined by

$$B(u,v):=\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx$$

is continuous and coercive, while the linear functional

$$L(v):=\int_\Omega f(x)v(x)\,dx+\int_{\partial\Omega} g(\xi)v(\xi)\,d\mathcal{H}^{n-1}(\xi)$$

is continuous from H to $\mathbb{R}$ by continuity of the trace operator. □

Exercise 3. Let $\Omega\subset\mathbb{R}^n$ be a bounded open set with smooth boundary, with $\partial\Omega=\Gamma_1\cup\Gamma_2$, and assume that $\mathcal{H}^{n-1}(\Gamma_1)>0$. Let $f\in L^2(\Omega)$. Find the weak formulation of the mixed boundary-value problem

$$\begin{cases}-\Delta u=f \text{ in } \Omega\\ u=0 \text{ on } \Gamma_1\\ \frac{\partial u}{\partial\nu}=0 \text{ on } \Gamma_2\end{cases} \tag{11.14}$$

and prove existence of a weak solution by using Lax-Milgram's theorem in a suitable Hilbert subspace of $H^1(\Omega)$.

Solution: The presence of the Dirichlet boundary condition suggests to operate in the Hilbert subspace

$$H := \{u \in H^1(\Omega), u = 0 \text{ on } \Gamma_1\},$$

where we have already seen that the Poincaré inequality

$$\|\nabla u\|_2 \geq C_\Omega \|u\|_2$$

holds (short reminder: the only constant function in H is 0 and $\partial\Omega$ is smooth).

The weak formulation is: finding $u \in H$ such that

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \int_\Omega f(x)v(x)\, dx \tag{11.15}$$

for every $v \in H$. Indeed, any smooth solution of (11.14) satisfies (11.15). Conversely, if $u \in H$ satisfies (11.15) and additionally $u \in H^2(\Omega)$, testing (11.15) on C_c^∞ functions gives $-\Delta u = f$ in $L^2(\Omega)$. Using this additional information, integrating by parts and using again (11.15) we get for every $v \in H$

$$\begin{gathered}\int_\Omega f(x)v(x)\, dx = \\ -\int_\Omega \Delta u(x)v(x)\, dx = \int_\Omega \nabla u(x)\cdot\nabla v(x)\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu}(\xi)v(\xi)\, d\mathcal{H}^{n-1}(\xi) = \\ \int_\Omega f(x)v(x)\, dx - \int_{\Gamma_2} \frac{\partial u}{\partial \nu}(\xi)v(\xi)\, d\mathcal{H}^{n-1}(\xi),\end{gathered}$$

since $v = 0$ on Γ_1. This gives $\frac{\partial u}{\partial \nu} = 0$ in $L^2(\Gamma_2)$. Since $u \in H$ we also have $u = 0$ a.e. on Γ_1, so (11.15) is the weak formulation of (11.14).

Existence of a unique solution to (11.15) in H follows again by the Poincaré inequality and the Lax-Milgram Theorem. □

12. Exercises PDE 14.02.13

Exercise 1. *Differentiation of convolutions revisited.* Recall preliminarly the following form of the Jensen inequality: let $F \in L^\infty(\mathbb{R}^n), F \geq 0$ with

$$\int_{\mathbb{R}^n} F(y)\, dy = 1.$$

Then for every convex function $\varphi\colon \mathbb{R} \to \mathbb{R}$ and every $g \in L^1(\mathbb{R}^n)$ it holds

$$\varphi\Big(\int_{\mathbb{R}^n} g(y)F(y)\, dy\Big) \leq \int_{\mathbb{R}^n} \varphi(g(y))F(y)\, dy.$$

In the following ϱ_ε is a positive symmetric C^∞ mollifier with

$$\int_{\mathbb{R}^n} \varrho_\varepsilon(y)\, dy = 1 \tag{12.1}$$

and $\mathrm{supp}\varrho_\varepsilon \subset B(0,\varepsilon)$.

(a) Let $1 \leq p < \infty$ and $f \in W^{1,p}(\mathbb{R}^n)$. Prove that $D(\varrho_\varepsilon \star f) = \varrho_\varepsilon \star Df$ in $L^p(\mathbb{R}^n)$.

(b) Let $1 \leq p < \infty$ and $f \in W^{1,p}(\mathbb{R}^n)$. Prove that $\|\varrho_\varepsilon \star f\|_{W^{1,p}(\mathbb{R}^n)} \leq \|f\|_{W^{1,p}(\mathbb{R}^n)}$.

(c) Let Ω be a bounded open subset of $\mathbb{R}^n$ and $f \in W_0^{1,1}(\Omega)$. Prove that $D(\varrho_\varepsilon \star f) = \varrho_\varepsilon \star Df$ in $L^1(\Omega)$. *Hint*: extend f by setting it equal to 0 outside Ω. Is this extension in $W^{1,1}(\mathbb{R}^n)$?

(d) Let Ω be a bounded open subset of $\mathbb{R}^n$ and $f \in W^{1,1}(\Omega)$. Define

$$\Omega_\varepsilon := \{x \in \Omega : \quad \text{dist}\,(x, \partial\Omega) \geq \varepsilon\}.$$

Let $\Omega' \subset\subset \Omega_\varepsilon$. Prove that $D(\varrho_\varepsilon \star f) = \varrho_\varepsilon \star Df$ in $L^1(\Omega')$. *Hint*: try first with C^∞ functions, then approximate.

Solution*:* (a) Let $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R}^n)$. Using the properties of the convolution with the mollifiers and the definition of weak gradient we have

$$\int_{\mathbb{R}^n} \varrho_\varepsilon \star f(x)\,\mathrm{div}\,\varphi(x)\,dx = \int_{\mathbb{R}^n} f(x)\,\varrho_\varepsilon \star \mathrm{div}\,\varphi(x)\,dx =$$
$$\int_{\mathbb{R}^n} f(x)\,\mathrm{div}\,(\varrho_\varepsilon \star \varphi)(x)\,dx = -\int_{\mathbb{R}^n} Df(x)\cdot\,\varrho_\varepsilon \star \varphi(x)\,dx = -\int_{\mathbb{R}^n} \varrho_\varepsilon \star Df(x)\cdot\,\varphi(x)\,dx$$

as required.

(b) Applying Jensen's inequality with $F(y) = \varrho_\varepsilon(x-y)$ for fixed x and using Fubini's Theorem and (12.1), we get

$$\int_{\mathbb{R}^n} |\varrho_\varepsilon \star f(x)|^p\,dx = \int_{\mathbb{R}^n} \Big|\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y) f(y)\,dy\Big|^p\,dx \le$$
$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varrho_\varepsilon(x-y)|f(y)|^p\,dy\,dx = \int_{\mathbb{R}^n} |f(y)|^p\,dy$$

which proves $\|\varrho_\varepsilon \star f\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)}$. The same argument with Df in place of f gives the claim.

(c) Let $\tilde f$ be the extension of f simply obtained by $\tilde f = 0$ out of Ω. Trivially $\tilde f \in W^{1,1}(\mathbb{R}^n)$. Indeed, since $f \in W_0^{1,1}(\Omega)$ there is a sequence $f_n \in C_c^\infty(\Omega)$ converging to f in $W^{1,1}$. The extensions $\tilde f_n$ simply obtained by $\tilde f_n = 0$ out of Ω form now easily a Cauchy sequence in $W^{1,1}(\mathbb{R}^n)$ with limit $\tilde f$. Step (a) gives now $D(\varrho_\varepsilon \star \tilde f) = \varrho_\varepsilon \star D\tilde f$. Since $D\tilde f = Df$ in Ω and $D\tilde f = 0$ outside, we conclude.

(d) Let us only see the proof when $f \in C^\infty(\Omega)$ leaving to the reader the easy approximation argument. Since $\mathrm{supp}\varrho_\varepsilon \subset B(0,\varepsilon)$ and by symmetry of the convolution

$$\varrho_\varepsilon \star f(x) = \int_{\mathbb{R}^n} \varrho_\varepsilon(x-y) f(y)\,dy = \int_{\mathbb{R}^n} \varrho_\varepsilon(y) f(x-y)\,dy = \int_{B(0,\varepsilon)} \varrho_\varepsilon(y) f(x-y)\,dy\,.$$

Now, when $x \in \Omega'$, the gradient $Df(x-y)$ (in the variable x) exists for every $y \in B(0,\varepsilon)$, since in this case $x - y \in \Omega$ where f is differentiable. We can therefore derive under the sign of integral, getting the required equality. □

Exercise 2. Let $N = 2$ and $B(0,1) \subset \mathbb{R}^2$ be the unit ball in the plane.

(a) For $0 < r < 1$ let u_r be the unique solution of the problem

$$\begin{cases} \Delta u_r = 0 \text{ in } B(0,1) \setminus B(0,r) \\ u_r(x) = 1 \text{ if } |x| = 1 \\ u_r(x) = 0 \text{ if } |x| = r\,. \end{cases}$$

Compute u_r.

(b) Define

$$v_r(x) := \begin{cases} u_r(x) \text{ in } B(0,1) \setminus B(0,r) \\ 0 \text{ if } |x| \le r\,. \end{cases}$$

Prove that $v_r \in H^1(B(0,1))$ and that

$$\|v_r - 1\|_{H^1} \to 0$$

when $r \to 0$.

(c) Let $f \in C^1(\overline{B(0,1)})$. Construct a sequence f_r with $\mathrm{supp} f_r \subset \overline{B(0,1)}\setminus B(0,r)$ and $\|f_r - f\|_{H^1} \to 0$.

(d) Deduce that $H_0^1(B(0,1)\setminus\{0\}) = H_0^1(B(0,1))$.

(e) Let $N = 1$. Is that true that $H_0^1((-1,1)\setminus\{0\}) = H_0^1((-1,1))$? *Hint*: Sobolev convergence is particularly strong in dimension 1.

Solution*:* (a) The radial symmetry of the domain and of the data suggests a radial solution. Arguing in the same way used to construct the fundamental solution and imposing the boundary conditions one gets

$$u_r(x) = \frac{\log(|x|)}{|\log r|} + 1\,.$$

(b) v_r are indeed Lipschitz continuous since obtained by gluing together two C^1 functions with the same trace on the interface (see Exercise 2, 9-11.01.13). Furthermore, using polar coordinates

$$\int_{B(0,1)} |v_r(x) - 1|^2 \, dx = |B(0,r)| + \frac{2\pi}{\log^2 r} \int_r^1 \varrho \log^2 \varrho \, d\varrho \leq |B(0,r)| + \frac{2C\pi}{\log^2 r} \, .$$

Indeed, the function $\varrho \to \varrho \log^2 \varrho$ is uniformly bounded in $(0,1]$ since it has a limit (namely 0) when $\varrho \to 0$. Now, the right-hand side vanishes when $r \to 0$. About the first derivatives, again by a direct computation and using polar coordinates

$$\int_{B(0,1)} |D(v_r(x) - 1)|^2 \, dx = \int_{B(0,1)\setminus B(0,r)} |Du_r(x)|^2 \, dx = \frac{2\pi}{\log^2 r} \int_r^1 \frac{1}{\varrho} \, d\varrho = \frac{2\pi}{|\log r|}$$

which again vanishes when $r \to 0$. Thus, the claim is proved.

(c) Set $f_r := v_r f$. Clearly $\mathrm{supp} f_r \subset \overline{B(0,1)} \setminus B(0,r)$. Furthermore, since f is C^1 up to the boundary, we easily have

$$\|f_r - f\|_{H^1} \leq \|f\|_{C^1} \|v_r - 1\|_{H^1}$$

which gives the claim.

(d) We need to use a double approximation argument. First, of all, observe that clearly

$$C_c^\infty(B(0,1\setminus\{0\}) \subset H_0^1(B(0,1))\,;$$

taking the closure at the left-hand side, the inclusion $H_0^1(B(0,1)\setminus\{0\}) \subset H_0^1(B(0,1))$ is trivial. Now, let $f \in H_0^1(B(0,1))$ and let $f_n \in C_c^\infty(B(0,1))$ a sequence H^1-converging to f. By step (c) applied to each f_n, for every n we can construct $g_n \in H_0^1(B(0,1)\setminus\{0\})$ such that $\|g_n - f_n\|_{H^1} \leq \frac{1}{n}$ (construct a sequence approximating f_n and then choose a sufficiently close element of the sequence). Now, easily g_n converges to f in H^1! Since $H_0^1(B(0,1)\setminus\{0\})$ is a Banach space, $f \in H_0^1(B(0,1)\setminus\{0\})$, and this proves the reverse inclusion.

(e) No. If the equality holds, any $f \in H_0^1((-1,1))$ could be approximated by a sequence $f_n \in C_c^\infty((-1,1)\setminus\{0\})$ in the H^1 topology, and in particular, by Morrey's imbedding in dimension 1, uniformly. This would imply $f(0) = 0$ which has no general reason to be.

□

Exercise 3. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded connected open set, and $u \in C^\infty(\overline{\Omega})$ a solution of the elliptic equation $Lu = -\sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u = 0$ in Ω. Assume that the coefficients $a_{ij} \in C^1(\overline{\Omega})$. Show that for λ sufficiently large not depending on u the function $v := |\nabla u|^2 + \lambda u^2$ satisfies $Lv \leq 0$ in Ω. Deduce that

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\Big(\|\nabla u\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)}\Big)\,. \tag{12.2}$$

Solution*:* We called $A(x)$ the matrix whose coefficients are $a_{ij}(x)$ and we denote with $\langle\cdot,\cdot\rangle$ the scalar product between vectors in $\mathbb{R}^n$. The norm we consider on matrices is the Frobenius norm.

We start by observing that by a direct computation, for any $w \in C^\infty(\overline{\Omega})$ one has

$$Lw^2(x) = -\sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} w^2(x) = -\langle A(x)\nabla w(x), \nabla w(x)\rangle + 2w(x)Lw(x)\,.$$

For $w = \sqrt{\lambda} u$, since u is a solution and by ellipticity, we get

$$L(\lambda u^2(x)) \leq -\lambda\alpha |\nabla u(x)|^2 \tag{12.3}$$

with α the coercivity constant of A. For $k = 1,\dots,N$ and $w = \frac{\partial u}{\partial x_k}$ we have

$$L\Big(\Big(\frac{\partial u}{\partial x_k}\Big)^2(x)\Big) \leq -\alpha \Big|\nabla(\frac{\partial u}{\partial x_k})(x)\Big|^2 + 2\frac{\partial u}{\partial x_k}(x) L\Big(\frac{\partial u}{\partial x_k}(x)\Big)\,.$$

By a direct computation and since $Lu = 0$ we have

$$L\Big(\frac{\partial u}{\partial x_k}(x)\Big) = \frac{\partial}{\partial x_k} Lu(x) + \sum_{i,j=1}^N \frac{\partial}{\partial x_k} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(x) = \sum_{i,j=1}^N \frac{\partial}{\partial x_k} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} u(x)\,.$$

Since $a_{ij} \in C^1(\overline{\Omega})$ we get easily

$$2\frac{\partial u}{\partial x_k}(x)L\Big(\frac{\partial u}{\partial x_k}(x)\Big) \le C_N|\nabla u(x)||D^2u(x)|$$

with D^2u the Hessian matrix and C_N a constant depending on the C^1 norm of the a_{ij}'s and on the dimension N. Summing over k from 1 to N we have

$$L(|\nabla u|^2(x)) \le -\alpha|D^2u(x)|^2 + C_N|\nabla u(x)||D^2u(x)|\,. \tag{12.4}$$

We now recall Young's inequality in the following form: for every $a, b \in \mathbb{R}$ and $\varepsilon > 0$ one has

$$ab \le \frac{\varepsilon^2}{2}a^2 + \frac{1}{2\varepsilon^2}b^2\,. \tag{12.5}$$

Using now (12.3), (12.4) and (12.5) with $a = |D^2u(x)|$, $b = |\nabla u(x)|$ and $\varepsilon = \sqrt{\frac{\alpha}{2C_N}}$ we obtain

$$L(|\nabla u|^2(x) + \lambda u^2(x)) \le -\lambda\alpha|\nabla u(x)|^2 + \frac{4C_N^2}{\alpha}|\nabla u(x)|^2\,.$$

For $\lambda \ge \big(\frac{2C_N}{\alpha}\big)^2$ the right-hand side is nonpositive so that setting $v := |\nabla u|^2 + \lambda u^2$ one has $Lv \le 0$ in Ω. By the weak maximum principle

$$\|v\|_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\partial\Omega)}\,.$$

Now, trivially $\|\nabla u\|^2_{L^\infty(\Omega)} \le \|v\|_{L^\infty(\Omega)}$. It is also easy to se that there is a constant C depending on N and λ, thus on N, C_N and α such that $\|v\|_{L^\infty(\partial\Omega)} \le C\Big(\|\nabla u\|_{L^\infty(\partial\Omega)} + \|u\|_{L^\infty(\partial\Omega)}\Big)^2$. From this, (12.2) follows.

□