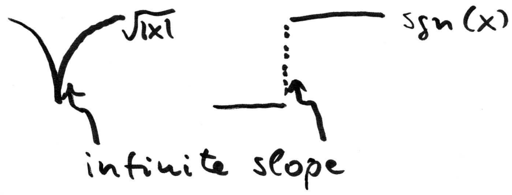
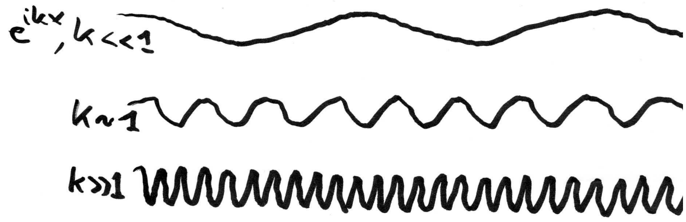
6 Regularity and the size of Fourier coefficients

Given a function $f \in L^2(-\pi, \pi)$, recall its Fourier coefficients $\widehat{f}_k = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx$. The Riemann-Lebesgue lemma says that the Fourier coefficients converge to zero as $|k| \to \infty$, but how fast? The answer depends on how smooth f is.

A vague intuition for this connection smoothness $\longleftrightarrow$ Fourier decay comes from the idea of *large gradients*. At a typical singularity where a function fails to be smooth, such as a jump or a cusp (see the piecture below), it has (infinitely) large gradient. On the other hand, the only Fourier modes with large gradients are the high ones (i.e. those with large $|k|$), since $\max_x |\frac{d}{dx} e^{ikx}| = |k|$.





This suggests that singular functions should contain a larger amount of high Fourier modes than smooth functions.

Our goal in this section is to make this connection quantitative. To this end we introduce for $n \in \mathbb{N} \cup \{0\}$:

$C^n_{per}(\mathbb{R}) := \{f \,:\, \mathbb{R} \to \mathbb{C} \;\mid\; f \ 2\pi\text{-periodic},$

f n times continuously differentiable$\}$.

In the sequel we abbreviate this space by C^n.

Theorem 1.7 *For any* $n \in \mathbb{N} \cup \{0\}$,

$$f \in C^n \implies |\widehat{f}_k| \le \frac{M}{|k|^n} \text{ for all } k,$$

where $M = \sup_x |\frac{d^n}{dx^n} f(x)|$.

Proof The idea is to integrate by parts in the definition of the Fourier coefficients. By periodicity of the integrand, the boundary contributions at π and $-\pi$ cancel and we obtain

$$\widehat{f}_k = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-ikx}}{-ik} f'(x)\, dx = \frac{1}{ik} \widehat{(f')}_k. \tag{43}$$

Iterating gives

$$\widehat{f}_k = \frac{1}{(ik)^n} \widehat{(\tfrac{d^n}{dx^n} f)}_k = \frac{1}{2\pi (ik)^n} \int_{-\pi}^{\pi} e^{-ikx} \tfrac{d^n}{dx^n} f(x) dx. \tag{44}$$

Bounding the absolute value of the last integral by the supremum of the absolute value of the integrand times the length of the interval of integration yields the assertion.

This result has a beautiful generalization to functions “inbetween” C^0 and C^1. For any function $f : \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{R}^n$, define its **modulus of continuity** by

$$\omega(\delta) := \sup_{x,y \in \Omega,\, |x-y| < \delta} |f(x) - f(y)| \quad (\delta > 0).$$

Example 1 Uniform continuity of f on Ω (and hence, in the case when Ω is closed and bounded, continuity of f) is equivalent to the statement that

$$\omega(\delta) \to 0 \text{ as } \delta \to 0.$$

Example 2 If f is continuously differentiable, then $|f(x) - f(y)| = |\int_{\min\{x,y\}}^{\max\{x,y\}} f'(s)ds| \leq M|x-y|$, $M = \sup_{s\in\Omega}|f'(s)|$, whence

$$\omega(\delta) \leq M\delta,$$

i.e. the modulus of continuity is bounded by a linear function of δ.

Example 3 If f is Hölder continuous with exponent $\alpha \in (0,1)$, then by definition there exists a constant $M > 0$ such that $|f(x) - f(y)| \leq M|x-y|^\alpha$ for all $x, y \in \Omega$, so

$$\omega(\delta) \leq M\delta^\alpha,$$

i.e. the modulus of continuity is bounded by a power of δ.

Theorem 1.8 *(Zygmund's theorem) If $f : \mathbb{R} \to \mathbb{C}$ is continuous and 2π-periodic,*

$$|\widehat{f}_k| \leq \frac{1}{2}\omega\left(\frac{\pi}{|k|}\right) \quad \textit{for all } k.$$

Corollary 1.1 *If $f : \mathbb{R} \to \mathbb{C}$ is 2π-periodic and Hölder continuous with exponent α, then*

$$|\widehat{f}_k| \leq \frac{C}{|k|^\alpha},$$

where $C = \pi^\alpha M/2$ and M is the constant from the definition of Hölder continuity.

Proof of the corollary This is immediate from the theorem and the estimate for $\omega(\delta)$ in Example 3.

As another corollary, for f as in Theorem 1.8 we obtain:

New proof of the Riemann-Lebesgue lemma By the assumptions on f, f is uniformly continuous, so $\omega(\delta) \to 0$

as $\delta \to 0$ by Example 1, whence $|\widehat{f}_k| \to 0$ as $|k| \to \infty$ by Theorem 1.8.

Proof of Theorem 1.8 The proof is a real gem: spectacularly short even though the result is very remarkable.

Because the integrand in the definition of the Fourier coefficient $\widehat{f}_k$ is periodic, we have

$$\widehat{f}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\tfrac{\pi}{k})\underbrace{e^{-ik(x+\frac{\pi}{k})}}_{=-e^{-ikx}}dx.$$

Take the average of these two expressions for $\widehat{f}_k$. It is still equal to $\widehat{f}_k$, and so

$$\widehat{f}_k = \frac{1}{4\pi}\int_{-\pi}^{\pi} \left(f(x) - f(x+\tfrac{\pi}{k})\right) e^{-ikx}dx.$$

The absolute value of the integrand is $\leq \omega(\frac{\pi}{|k|})$, and so

$$|\widehat{f}_k| \leq \frac{1}{4\pi}\cdot 2\pi \cdot \omega\left(\frac{\pi}{|k|}\right),$$

as asserted.

Now we show that conversely, if the Fourier coefficients of a function are sufficiently rapidly decaying, then the function is smooth.

Theorem 1.9 *Let $f \in L^2(-\pi,\pi)$, $n \in \mathbb{N}\cup\{0\}$. Then*

$$|\widehat{f}_k| \leq \frac{C}{|k|^{n+1+\epsilon}} \text{ for some } C>0,\, \epsilon>0 \implies f\in C^n.$$

Technical remark The statement $f \in C^n$ in the theorem is (customary) shorthand for: there exists a function $\tilde{f} \in C^n_{per}(\mathbb{R})$ such that $f = \tilde{f}$ a.e. in $(-\pi,\pi)$. Without this freedom to change f on a set of measure zero the above result can of course not be correct: such changes don't change the Fourier coefficients (as these are integrals) but can make any function discontinuous. For example the infinitely differentiable function $f = 0$ has the same Fourier coefficients as the discontinuous L^2 function $f(x) = 0$ for $x \neq 0$, $f(x) = 1$ at $x = 0$.

Proof The result is nontrivial and interesting even when $n = 0$: if f is an L^2 function whose Fourier coefficients decay like $|k|^{-(1+\epsilon)}$ then f is already continuous. So let us start with this case. We split the proof into three steps:

Claim 1 $S_{N,f}$ is absolutely and uniformly convergent.
Claim 2 The limit $\tilde{f}$ of $S_{N,f}$ is continuous.
Claim 3 $\tilde{f} = f$ a.e.

To show the first claim, note that for $M > N$, by the decay hypothesis on the $\widehat{f}_k$

$$\begin{aligned} |S_{M,f}(x) - S_{N,f}(x)| &= \Big| \sum_{N<|k|\leq M} \widehat{f}_k e^{ikx} \Big| \\ &\leq \sum_{N<|k|\leq M} \frac{C}{|k|^{1+\epsilon}} \leq \sum_{|k|=N+1}^{\infty} \frac{C}{|k|^{1+\epsilon}} \end{aligned}$$

which, due to the assumption $\epsilon > 0$, is a tailsum of a convergent series, and tends to zero as $N \to \infty$. Thus $S_{N,f}(x)$ is a uniform Cauchy sequence, and hence uniformly convergent.

The second claim now follows from the general fact that the uniform limit of a sequence of continuous functions (here: the $S_{N,f}(x)$) is continuous.

The third claim is less elementary. We will use the L^2 theory of Fourier series (Theorem 1.3) to infer $||f - \tilde{f}||_{L^2} = 0$. Indeed,

$$||f - \tilde{f}||_{L^2} \leq ||f - S_{N,f}||_{L^2} + ||S_{N,f} - \tilde{f}||_{L^2}$$

and the first term on the right tends to zero as $N \to \infty$ by Theorem 1.3, and the second term tends to zero because $S_{N,f} - \tilde{f}$ tends to zero uniformly by Step 1 and because uniform convergence implies L^2 convergence (see auxiliary statement 2) in the proof of Theorem 1.3). This completes the proof of Theorem 1.9 in case $n = 0$.

Now let $n = 1$. By the result for $n = 0$, f is continuous and its partial Fourier sums $S_{N,f}$ converge uniformly to f. We now look at the termwise differentiated series

$$\sum_{k \in \mathbb{Z}} \underbrace{ik\widehat{f}_k e^{ikx}}_{=:h_k(x)}, \tag{45}$$

and appeal to the following general principle from first year analysis: if a series of smooth functions $h_k(x)$ and the termwise differentiated series both converge absolutely and uniformly on some open set, then the original series is differentiable and its derivative equals the termwise derivative. Since by assumption $|\widehat{f}_k| \le C/|k|^{2+\epsilon}$, the h_k satisfy

$$|h_k| \le \frac{C}{|k|^{1+\epsilon}},$$

and so the series (45) is absolutely and uniformly convergent as required, and moreover the limit is continuous. Thus f is continuously differentiable.

Finally, the assertion for $n > 1$ follows by applying the argument for $n = 1$ successively to f', f'', etc. This establishes the theorem.

Let us summarize the situation as described by Theorems 1.7 and 1.9:

$$\begin{aligned} |\widehat{f}_k| &\le \tfrac{C}{|k|^{n+1+\epsilon}} \Longrightarrow f \in C^n \\ |\widehat{f}_k| &\le \tfrac{C}{|k|^{n}} \quad\;\; \Longleftarrow f \in C^n. \end{aligned}$$

Mathematical interpretation Smoothness properties of functions can be studied fairly precisely by investigating the decay of their Fourier coefficients. This is very useful, for example, in the study of regularity of solutions to partial differential equations (see Section 5).

Data compression interpretation Smooth functions can be efficiently stored and reconstructed by keeping only its first few Fourier coefficients. For rough functions, however, this is not a good idea, as a large amount of information sits in the high Fourier coefficients.

Mind the gap The different decay rates in the two statements are not caused by the statements not being optimal (in fact they are, see Problem 1 below). It is related to a basic structural mismatch: being C^n is a pointwise property of f, but the individual Fourier coefficients – being integrals – only reflect the "average" behaviour of f. Likewise, the above decay bounds on the $\widehat{f}_k$ are pointwise in k, but the differentiability behaviour of f – since f is a sum of contributions over many k – corresponds to a type of "average" behaviour of the $\widehat{f}_k$.

Outlook: An exact equivalence between smoothness and Fourier decay

With a little more work, it is possible to formulate an exact equivalence between smoothness and Fourier decay, by measuring both in a suitable average rather than pointwise sense. We begin by introducing the following "average" notion of differentiability, in which the requirement of pointwise convergence of difference quotients is replaced by L^2-convergence.

Definition Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic, $f \in L^2(-\pi, \pi)$. We say that f has a **weak derivative** $g \in L^2(-\pi, \pi)$ if there exists $g \in L^2(-\pi,\pi)$ such that

$$||\tfrac{f(\cdot+h)-f}{h} - g||_{L^2} = \left(\int_{-\pi}^{\pi} |\tfrac{f(x+h)-f(x)}{h} - g(x)|^2 dx\right) \to 0 \text{ as } h \to 0.$$

Such a function g, if it exists, is unique and is denoted f' or $\frac{d}{dx}f$.[19]

We now state without proof:

Theorem 1.10 *Let $f : \mathbb{R} \to \mathbb{C}$ be 2π-periodic, $f \in L^1(-\pi,\pi)$.*

a) $f \in L^2(-\pi,\pi) \Longleftrightarrow \sum_{k\in\mathbb{Z}} |\widehat{f}_k|^2 < \infty$

b) $f \in L^2(-\pi,\pi)$, f has weak derivatives up to order s in $L^2(-\pi,\pi) \Longleftrightarrow \sum_{k\in\mathbb{Z}} \Big((1+|k|^s)|\widehat{f}_k|\Big)^2 < \infty$.

The main point in the proof is to establish that if f is s times weakly differentiable, then the Fourier coefficients of the s^{th} weak derivative are given by the same expression as for classical s^{th} derivatives of a smooth function found in (44), i.e.

$$(\widehat{\tfrac{d^s}{dx^s}f})_k = (ik)^s \widehat{f}_k.$$

This together with Parseval's equation implies that

$$||\tfrac{d^s}{dx^s}f||_{L^2}^2 = 2\pi \sum_{k\in\mathbb{Z}} |k|^{2s}|\widehat{f}_k|^2.$$

The set of functions satisfying either (and hence both) properties of Theorem 1.10 b) is a basic example of a so-called *Sobolev space*[20], and is usually denoted $H^s_{per}(\mathbb{R})$ or $W^{s,2}_{per}(\mathbb{R})$. In fact, these spaces are well-defined also for non-integer $s \geq 0$, via the second property in b) which continues to make sense.

[19] In textbooks one usually finds an equivalent but more complicated definition of "weak derivative", which – unlike the above – can be generalized even to functions not belonging to L^2 or L^1, such as the "delta function". See Chapter III.

[20] these are spaces of functions with a finite number of weak derivatives belonging to L^p, and can be defined on arbitrary open subsets of $\mathbb{R}^n$

Such spaces – which emerged here from our desire to design an exact match between order of differentiability and decay rate of Fourier coefficients – also emerge, for different reasons, in the theory of partial differential equations, where they play a very important role.

What is the relationship of these new function spaces with classical Fourier decay and classical differentiability as appearing in Theorems 1.7 and 1.9? By the trivial inequality $|\widehat{f}_k|\,|k|^s \le (\sum_{\ell\in\mathbb{Z}} |\widehat{f}_\ell|^2(1+|\ell|^{2s}))^{1/2}$,

$$f \in H^s_{per}(\mathbb{R}) \quad \Longrightarrow \quad |\widehat{f}_k| \le \frac{C}{|k|^s}.$$

Moreover, combining this with theorem 1.9 yields the implication

$$f \in H^s_{per},\ s > n+1 \quad \Longrightarrow \quad f \in C^n_{per}.$$

In other words, a sufficiently high order of weak differentiability implies classical differentiability! This statement is a special[21] case of an important and much more general result, the so-called *Sobolev embedding theorem* (see e.g. [Ev]).

[21]and nonoptimal, $s > n + \frac{1}{2}$ would suffice. To see this, you estimate $\sum_k |\widehat{f}_k|(1+|k|)^n = \sum_k |\widehat{f}_k|(1+|k|)^s(1+|k|)^{-(s-n)} \le (\sum_k |\widehat{f}_k|^2(1+|k|)^{2s})^{1/2}(\sum_k(1+|k|)^{-2(s-n)})^{1/2}$ by the Cauchy-Schwarz inequality, note that the second term is a convergent sum when $s - n > \frac{1}{2}$, and show by arguing similarly to the proof of Theorem 1.9 that convergence of the sum on the left implies that f is C^n.

Problems

1. (More about regularity and decay of Fourier coefficients)

   For $n = 0$ and $n = 1$, show that the assumption in Theorem 1.9 that

   $$|\hat{f}_k| \leq \frac{C}{|k|^{n+1+\epsilon}} \text{ for some } \epsilon > 0$$

   cannot be weakened to

   $$|\hat{f}_k| \leq \frac{C}{|k|^{n+1}},$$

   by giving an example of a function in $L^2(-\pi, \pi)$ which satisfies the second decay condition but fails to be C^n. Hint: It suffices to search among functions which have appeared already in these notes.

Notes The material in this section is standard, except for Theorem 1.8, which can be found in [Zy].