3 The Fourier transform on $\mathbf{L^2}$

The function space we have used so far,

$$X := \{f : \mathbb{R}^n \to \mathbb{C} \mid f \in L^1(\mathbb{R}^n),\ \widehat{f} \in L^1(\mathbb{R}^n)\}, \qquad (64)$$

is a natural setting from a theoretical point of view: it is the largest space in which the integrals appearing in the definition of the Fourier transform and the reconstruction formula are well defined, i.e. in which the Fourier transform and its inverse make sense as "integral transformations".

But for applications this setting turns out to be too restrictive. When solving wave equations (see Section 6.4), one would like to employ Fourier calculus for the function $\frac{\sin x}{x}$, which is not in L^1, but only in L^2.[30] When solving Schrödinger equations (see Section 6.5), it would be useful to be able to Fourier transform the function $e^{it|x|^2}$ ($t \in \mathbb{R}$), which is not in L^1 but only in L^∞. From a theoretical standpoint, Plancherel's formula (Theorem 2.1 2)) as well as previous good experience with Fourier series (see Section 4) suggest to investigate the behaviour of the Fourier transform on L^2. The presence of the plane waves $e^{ik\cdot x}$ in (48) makes it natural to enquire as to their Fourier transform, but they do not belong to any L^p space other than L^∞.

This leads to the following

Question How do we define the Fourier transform of functions in $L^2(\mathbb{R}^n)$ or $L^\infty(\mathbb{R}^n)$?

As already explained, these spaces are not contained in L^1; so the basic definition (46) of the Fourier transform as an integral

[30] Note that on unbounded domains like $\mathbb{R}^n$, the spaces L^p ($1 < p \le \infty$) are not contained in L^1. Example of an L^p function not belonging to L^1: $f(x) = (1 + |x|)^{-n}$. For the definition of L^p and basic properties see Appendix D.

no longer makes sense.

Two different approaches have been developed to overcome this problem, based on

- Approximation by L^1 functions. This leads to the "L^2 theory" presented in this subsection.
- Duality. This approach is more difficult but more general; see Section 4.

Our construction of the Fourier transform on L^2 relies on the following

Definition A map $A : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is said to be **continuous** if for any sequence f_j which converges to f in L^2 (i.e. $||f_j - f||_2 \to 0$), $A(f_j)$ converges to $A(f)$ in L^2.

Theorem 2.3 *There exists a unique map $\mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ which*
(a) is continuous
(b) agrees with the Fourier transform on $L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$.
Moreover this map satisfies

$$||f||_2^2 = (2\pi)^{-n} ||\mathcal{F}f||_2^2 \text{ for all } f \in L^2(\mathbb{R}^n). \tag{65}$$

Definition Let $\mathcal{F}$ be the map from Theorem 2.3. For $f \in L^2(\mathbb{R}^n)$, its Fourier transform is defined by $\widehat{f} := \mathcal{F}f$.

Proof First we show existence. Let $f \in L^2$. If $f \in L^1 \cap L^2$, we define $\mathcal{F}f = \widehat{f}$, $\widehat{f}$ being the usual Fourier transform (46). If $f \notin L^1$, choose any sequence $f_j \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ such that $||f_j - f||_2 \to 0$. (For instance, take $f_j = \chi_{R_j} f$, where χ_R denotes the characteristic function of the ball around the origin of radius R, and the radii R_j are chosen to tend to ∞.) We will use the following lemma, whose proof we momentarily postpone.

Lemma 2.1 *Let $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\widehat{g} \in L^2(\mathbb{R}^n)$ and Plancherel's formula $||g||_2^2 = (2\pi)^{-n}||\widehat{g}||_2^2$ holds.*

We now have the following chain of implications

$$\begin{aligned}
& f_j \text{ convergent in } L^2 \\
\Longrightarrow\ & f_j \text{ a Cauchy sequence in } L^2 \\
\Longrightarrow\ & \widehat{f_j} \text{ a Cauchy sequence in } L^2 \text{ (by Lemma 2.1)} \\
\Longrightarrow\ & \widehat{f_j} \text{ convergent in } L^2 \text{ (by completeness of } L^2\text{)}.
\end{aligned}$$

Consequently there exists $v \in L^2$ such that $||\widehat{f_j} - v||_2 \to 0$. Set $\mathcal{F}f := v$.

By construction, the map $\mathcal{F}$ defined above satisfies (b). We claim that it also satisfies (65) (and hence in particular satisfies a)). Indeed, by Plancherel's formula applied to the f_j, $||f_j||_2^2 = (2\pi)^{-n}||\widehat{f_j}||_2^2$, but $f_j \to f$ and $\widehat{f_j} \to \mathcal{F}f$ in L^2 and $|| \cdot ||_2$ is continuous with respect to L^2 convergence, establishing (65).

It remains to show uniqueness. Suppose $\mathcal{F}^{(1)}, \mathcal{F}^{(2)} : L^2 \to L^2$ are continuous maps which agree with each other on $L^1 \cap L^2$. Let $f \in L^2$ be arbitrary. Since $L^1 \cap L^2$ is dense in L^2, we may pick a sequence $f_j \in L^1 \cap L^2$ with $f_j \to f$ in L^2. Then

$$\begin{array}{ccccl}
\mathcal{F}^{(1)} f_j & = & \mathcal{F}^{(2)} f_j & & \text{for all } j \\
\downarrow & & \downarrow & & \\
\mathcal{F}^{(1)} f & & \mathcal{F}^{(2)} f & &
\end{array}$$

with the arrows denoting convergence in L^2 as $j \to \infty$. Hence $\mathcal{F}^{(1)}f = \mathcal{F}^{(2)}f$.

From a more abstract point of view, the key ingredients of the above construction of the Fourier transform on L^2 were

– a dense subset of L^2 (in this case $L^2 \cap L^1$)

– continuity of the Fourier transform on the dense subset (guaranteed here by the validity of Plancherel's formula on $L^2 \cap L^1$).

In the same vein, we could have constructed the Fourier transform on L^2 by starting from the smaller space of Schwartz functions defined at the beginning of Section 4 (as is sometimes done in the literature), or we could have extended any continuous map from a dense subset of a complete metric space to the whole space. (For example, a continuous map defined on the rational points of an interval $[a, b]$ can be extended uniquely to a continuous map on $[a, b]$.)

Proof of Lemma 2.1 As in the proof of the reconstruction formula (48), we consider the convolution $G_\sigma * f$, G_σ being the Gaussian (54) with standard deviation $\sigma > 0$. Since G_σ and f belong to L^1, so does $G_\sigma * f$, and since $\widehat{G_\sigma}$ decays exponentially, $\widehat{G_\sigma * f} = \widehat{G_\sigma}\widehat{f}$ belongs to L^1. Hence by Plancherel's formula for L^1 functions with L^1 Fourier transform (Theorem 2.1 2)) and the explicit formula for the Fourier transform of a Gaussian (Example 2, Section 1)

$$\int_{\mathbb{R}^n} |G_\sigma * f|^2 = (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{G_\sigma * f}|^2 = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-\sigma^2|k|^2} |\widehat{f}(k)|^2 dk.$$

We now pass to the limit $\sigma \to 0$. Either by direct estimates, or by appealing to a general result on approximation of L^p functions (see Lemma 3.4 a) in Section 3.3), $G_\sigma * f \to f$ in $L^2(\mathbb{R}^n)$, so the left hand side converges to $\int_{\mathbb{R}^n} |f|^2$. And by the monotone convergence theorem[31], the integral on the right hand side converges to the integral of the pointwise limit $|\widehat{f}(k)|^2$ of the

[31] This basic result from integration theory states that if $g_n : \mathbb{R}^n \to \mathbb{R}$ is a sequence of measurable functions such that $0 \le g_1(x) \le g_2(x) \le \ldots$ for a.e. $x \in \mathbb{R}^n$, and $g_n(x) \to g(x)$ a.e., then g is measurable and $\int_{\mathbb{R}^n} g_n \to \int_{\mathbb{R}^n} g$ as $n \to \infty$. For a proof see e.g. [Ru], Chapter 1.

integrand. Hence $\widehat{f} \in L^2$ and Plancherel's formula holds.

Example As an elementary illustration of Lemma 2.1, let us evaluate the integral $I = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx$. We know from Example 1, Section 4 that

$$\widehat{\tfrac{1}{2}\chi_{[-1,1]}}(x) = \frac{\sin x}{x}.$$

The right hand side is not in L^1 (it is only in L^2), so we cannot apply Theorem 2.1. But $\frac{1}{2}\chi_{[-1,1]}$ is in $L^1 \cap L^2$, so the weaker hypothesis for Plancherel's formula given in Lemma 2.1 is satisfied, and so

$$I = 2\pi \int_{\mathbb{R}} (\frac{1}{2}\chi_{[-1,1]})^2 = \pi.$$