Chapter II

The Fourier Transform

1 The Fourier transform on $\mathbb{R}^n$

In Sections 1-3, we studied functions on the unit circle $S^1 = \mathbb{R} \bmod 2\pi$, and expanded them into trigonometric functions e^{ikx} with integer frequencies $k \in \mathbb{Z}$.

Now we consider functions on the real line $\mathbb{R}$, and expand them into trigonometric functions e^{ikx} with real frequencies $k \in \mathbb{R}$.

The remarkable phenomenon that the domain $\leftrightarrow$ frequency domain correspondences are

$$S^1 \leftrightarrow \mathbb{Z}$$

on the one hand and

$$\mathbb{R} \leftrightarrow \mathbb{R}$$

on the other hand has a beautiful group-theoretic explanation. See Section **??**.

We begin by introducing the Fourier transform of functions on $\mathbb{R}$, which replaces the notion of Fourier coefficients of periodic functions. In fact, without extra labour we may consider functions on $\mathbb{R}^n$. We then prove the corresponding representation formula, and develop a basic calculus showing how the Fourier transform behaves under operations such as multiplication, convolution and differentiation. Finally, to illustrate the power of this calculus we solve the heat equation in $\mathbb{R}^n$ (see Section 2) and derive the celebrated Heisenberg uncertainty principle (see Section **??**).

Definition (Function space L^1) Let $\Omega \subseteq \mathbb{R}^n$ be measurable. $L^1(\Omega) := \{f : \Omega \to \mathbb{C} \,|\, \text{f measurable}, ||f||_1 := \int_\Omega |f| < \infty\}$.

Definition For any function $f \in L^1(\mathbb{R}^n)$, define its **Fourier transform** as follows

$$\widehat{f}(k) := \int_{\mathbb{R}^n} e^{-ik\cdot x} f(x)\, dx \quad (k \in \mathbb{R}^n), \tag{46}$$

where $k \cdot x = \sum_{j=1}^n k_j x_j$ is the usual dot product between $x = (x_1, .., x_n) \in \mathbb{R}^n$ and $k = (k_1, .., k_n) \in \mathbb{R}^n$. Thus the Fourier transform $\widehat{f}$ is again a function from $\mathbb{R}^n$ to $\mathbb{C}$.

Theorem 2.1 *(Reconstruction formula; basic properties) Suppose $f : \mathbb{R}^n \to \mathbb{C}$ satisfies*

$$f,\, \widehat{f} \in L^1(\mathbb{R}^n). \tag{47}$$

Then f can be reconstructed from its Fourier transform as follows

$$f(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ik\cdot x} \widehat{f}(k)\, dk \text{ for a.e. } x \in \mathbb{R}^n. \tag{48}$$

Moreover, if g is another function satisfying (47), then fg, $f\overline{g}$, $\widehat{f}\widehat{\overline{g}}$, $f * g \in L^1(\mathbb{R}^n)$, and
1) $\int_{\mathbb{R}^n} f\,\overline{g} = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}\,\widehat{\overline{g}}$ (Plancherel's formula)
2) $\int_{\mathbb{R}^n} |f|^2 = (2\pi)^{-n} \int_{\mathbb{R}^n} |\widehat{f}|^2$ (Plancherel's formula)
3) $\widehat{f * g} = \widehat{f}\,\widehat{g}$ (F.T. of a convolution, see (49) below)
4) $\widehat{f\,g} = (2\pi)^{-n} \widehat{f} * \widehat{g}$ (F.T. of a product)
5) $\hat{\hat{f}}(x) = (2\pi)^n f(-x)$ for a.e. $x \in \mathbb{R}^n$ (F.T. of a F.T.)
6) $\widehat{f(\cdot + a)} = e^{ik\cdot a}\widehat{f}(k)$ (F.T. of a translate)
7) If in addition f is continuously differentiable and

$$\frac{\partial f}{\partial x_j} \in L^1(\mathbb{R}^n),$$

then $\widehat{\frac{\partial}{\partial x_j} f}(k) = ik_j \widehat{f}(k)$ (F.T. of a derivative).

Here the $f(\cdot + a)$ denotes the function g defined by $g(x) = f(x+a)$, and the convolution of two functions as appearing in 3) and 4) is defined as follows:

$$(f * g)(x) := \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy. \tag{49}$$

Note that if f, g belong to $L^1(\mathbb{R}^n)$, then by Fubini's theorem the integrand in (49) belongs to $L^1(\mathbb{R}^n)$ for a.e. x, so the right hand side is well defined, and $f * g$ lies again in $L^1(\mathbb{R}^n)$. See the beginning of the proof of 3) below.

For applications the key point is 7): the Fourier transform turns differentiation (with respect to x_j) into something much simpler, namely multiplication (by ik_j). This can be utilized to solve partial differential equations, and constituted Fourier's original motivation to introduce (46). See Section 2.

A subtle aspect of formula (48) not present in the simpler context of Fourier series is the fact that the natural space of functions f we wish to decompose, $L^1(\mathbb{R}^n)$, does *not* contain the functions $e^{ik\cdot x}$ into which we decompose them. Note that the latter are periodic functions of x when $n = 1$ (and periodic functions of x in direction $k/|k|$ and constant in orthogonal directions when $n > 1$), and so do not belong to $L^1(\mathbb{R}^n)$ and do not go to zero as $|x| \to \infty$.

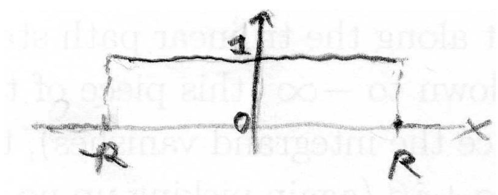
On the other hand, the reconstruction formula implies that their superposition (via integration over k) belongs to L^1 if f does, and goes to zero as $|x| \to \infty$ if f does. Thus a large amount of cancellation occurs between the different contributions $\widehat{f}(k)e^{-ik\cdot x}$ in (48).

We remark that in physics functions of form $f(x) = ae^{ik\cdot x}$, $a \in \mathbb{C}$, $k \in \mathbb{R}^n$, are called **plane waves**, with a and k being called, respectively, its **amplitude** and its **wavevector**.

Finally, the reader is warned that while our definition (46) of the Fourier transform is probably the most common one, some alternative definitions (with additional constants inserted at various places) continue to be in use in the literature. These are motivated by the authors' desire to make their favourite rule of Fourier calculus come out particularly "nice"; the drawback is that other rules will then become more complicated. For a short table see Appendix D.

A large list of Fourier transforms in one dimension that can be calculated explicity can be found in [Ob].

Example 1 (Fourier transform of a characteristic function) Let $n = 1$, $f(x) = \chi_{(-R,R)}(x)$, $R > 0$.

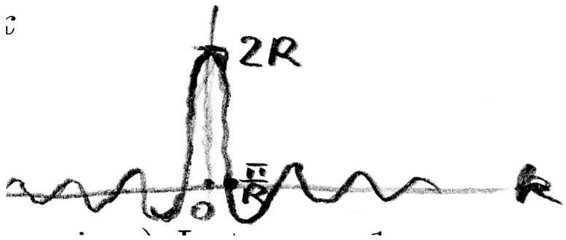


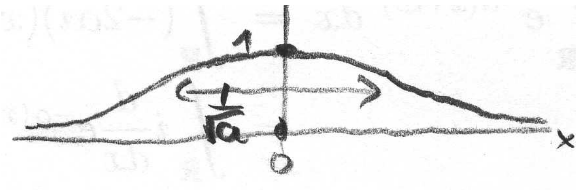
Calculate

$$\begin{aligned}\widehat{f}(k) &= \int_{\mathbb{R}} e^{-ikx}\chi_{(-R,R)}(x)\,dx = \int_{-R}^{R} e^{-ikx}dx \\ &= \frac{1}{-ik}e^{-ikx}\,\Big|_{-R}^{R} = -\frac{e^{-ikR}-e^{ikR}}{ik} \\ &= 2\frac{e^{ikR}-e^{-ikR}}{2ik} = 2\frac{\sin Rk}{k}.\end{aligned}$$

Example 2[22] (Fourier transform of a Gaussian) Let $n = 1$, $f(x) = e^{-ax^2/2}$, $a > 0$.

[22]This example will be useful in the proof of the reconstruction formula.





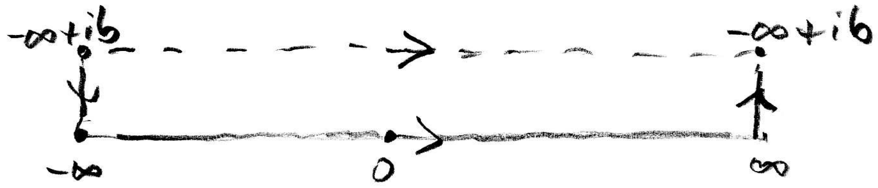
Calculate

$$
\begin{aligned}
\widehat{f}(k) &= \int_{\mathbb{R}} e^{-ax^2/2-ikx}dx \\
&= \int_{\mathbb{R}} e^{-(\sqrt{\frac{a}{2}}x+\frac{i}{\sqrt{2a}}k)^2+(\frac{i}{\sqrt{2a}}k)^2}dx \\
&= e^{-k^2/2a}\int_{\mathbb{R}} e^{-(\sqrt{\frac{a}{2}}x+\frac{i}{\sqrt{2a}}k)^2}dx. \qquad (50)
\end{aligned}
$$

We claim that for $\alpha > 0$, $\beta \in \mathbb{R}$, $\int_{\mathbb{R}} e^{-\alpha(x+i\beta)^2}dx$ is independent of β.

If you are familiar with contour integration, you can understand this as follows. The function $z \mapsto e^{-\alpha z^2}$ is holomorphic, and hence by Cauchy's theorem, integrating it along the linear path from $-\infty + i\beta$ across to $\infty + i\beta$ gives the same result as integrating it along the trilinear path starting from $-\infty + i\beta$, then going down to $-\infty$ (this piece of the path gives no contribution since the integrand vanishes), then across to ∞, and then up to $\infty + i\beta$ (again picking up no contribution since the integrand vanishes). See the picture below.

In case you are not familiar with contour integration, an



alternative proof can be given as follows:

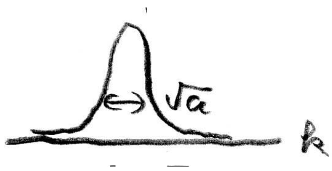
$$\begin{aligned}\frac{d}{d\beta}\int_{\mathbb{R}} e^{-\alpha(x+i\beta)^2}dx &= \int_{\mathbb{R}}(-2\alpha i)(x+i\beta)e^{-\alpha(x+i\beta)^2} \\ &= \int_{\mathbb{R}} i\frac{d}{dx}e^{-\alpha(x+i\beta)^2}dx \\ &= ie^{-\alpha(x+i\beta)^2}\Big|_{x=-\infty}^{\infty} = 0.\end{aligned}$$

This fact together with eq. (50), the substitution $y = \sqrt{\frac{a}{2}}x$, and the fact that

$$\int_{\mathbb{R}} e^{-y^2}dy = \sqrt{\pi} \tag{51}$$

yields

$$\begin{aligned}\widehat{f}(k) &= e^{-\frac{k^2}{2a}}\int_{\mathbb{R}} e^{-(\sqrt{\frac{a}{2}}x)^2}dx = \sqrt{\frac{2}{a}}e^{-\frac{k^2}{2a}}\int_{\mathbb{R}} e^{-y^2}dy \qquad (52) \\ &= \sqrt{\frac{2\pi}{a}}e^{-\frac{k^2}{2a}}.\end{aligned}$$



Thus up to normalization factors, the Fourier transform of a Gaussian with standard deviation $\frac{1}{\sqrt{a}}$ is a Gaussian with standard deviation $\sqrt{a}$ (Fourier, 1811). The fact that the standard deviations are inverse to one another is a special case of a remarkable general result, the Heisenberg uncertainty principle, discussed in Section **??**.

We remark that an elegant but little known way to evaluate the Gaussian integral (51) is to use Fourier calculus. For $a = 1$, the above calculation (52) shows that $\widehat{f} = \sqrt{2}Cf$ with $C = \int_{\mathbb{R}} e^{-y^2} dy$, and hence by Plancherel's formula

$$\int_{\mathbb{R}} |f|^2 = (2\pi)^{-1} \int_{\mathbb{R}} \underbrace{|\widehat{f}|^2}_{=2C^2|f|^2} ,$$

whence $1 = C^2/\pi$, and so $C = \sqrt{\pi}$.

Generalization of Example 2 to higher dimensions By the above result together with Fubini's theorem, if $f : \mathbb{R}^n \to \mathbb{R}$ is an n-dimensional Gaussian, $f(x) = e^{-a|x|^2/2}$, then

$$\begin{aligned} \widehat{f}(k) &= \int_{\mathbb{R}^n} e^{-ik\cdot x} f(x)\, dx \\ &= \underbrace{\left(\int_{\mathbb{R}} e^{-ik_1x_1} e^{-ax_1^2/2} dx_1\right)}_{=\sqrt{\frac{2\pi}{a}}e^{-k_1^2/(2a)}} \cdots \underbrace{\left(\int_{\mathbb{R}} e^{-ik_nx_n} e^{-ax_n^2/2} dx_n\right)}_{=\sqrt{\frac{2\pi}{a}}e^{-k_n^2/(2a)}} \\ &= \left(\frac{2\pi}{a}\right)^{n/2} e^{-|k|^2/(2a)}. \end{aligned}$$

In particular, by letting $a = 1/\sigma^2$ respectively $a = \sigma^2$ we obtain the following formula for the Fourier transform and the double Fourier transform of a normalized Gaussian with standard deviation σ:

$$\begin{aligned} &G_\sigma(x) = \frac{1}{(2\pi\sigma^2)^{n/2}} e^{-|x|^2/2\sigma^2} \\ &\Longrightarrow \widehat{G_\sigma}(k) = e^{-\sigma^2|k|^2/2}, \quad \widehat{\widehat{G}_\sigma}(x) = (2\pi)^n G_\sigma(x). \quad (53) \end{aligned}$$

We postpone the proof of the representation formula and begin with the

Proof of properties 1)-7), assuming (48) Before proving 1), we first check that both sides are well defined, i.e. that the integrands belong to $L^1(\mathbb{R}^n)$. By (46), $|\widehat{f}(k)| \le ||f||_1$ for all k, so $\widehat{f}$ is bounded and hence $\widehat{f}\overline{\widehat{g}}$ – as a product of a bounded function and an L^1 function – belongs to $L^1(\mathbb{R}^n)$. Analogously, by using the reconstruction formula instead of (46), one deduces $|f(x)| \le (2\pi)^{-n}||\widehat{f}||_1$ a.e., whence $f\overline{g} \in L^1(\mathbb{R}^n)$. Now to establish 1), we start from the expression on the left hand side, substitute (48) for g, and use Fubini's theorem together with $\overline{e^{ik\cdot x}} = e^{-ik\cdot x}$ to infer

$$\begin{aligned}
\int_{\mathbb{R}^n} f\,\overline{g} &= (2\pi)^{-n} \int_{\mathbb{R}^n} f(x) \overline{\Big(\int_{\mathbb{R}^n} \widehat{g}(k) e^{ik\cdot x} dk \Big)} dx \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \underbrace{\Big(\int_{\mathbb{R}^n} f(x)\, e^{-ik\cdot x} dx \Big)}_{=\widehat{f}(k)} \overline{\widehat{g}(k)}\, dk.
\end{aligned}$$

2) follows immedidately from 1), by setting $g = f$.

Before showing 3), it is useful to first prove 6). By the definition of g and by changing variables $y = x + a$,

$$\widehat{g}(k) = \int_{\mathbb{R}^n} e^{-ik\cdot x} f(x+a)\, dx = \int_{\mathbb{R}^n} e^{-i(y-a)\cdot k} f(y)\, dy = e^{ia\cdot k} \widehat{f}(k).$$

Next, we show that if $f,\ g \in L^1(\mathbb{R}^n)$, then so is $f * g$; in particular the Fourier transform appearing on the left hand side of 3) is well defined. Indeed, by the triangle inequality and Fubini's theorem

$$||f * g||_1 = \int_{\mathbb{R}^n} \Big| \int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy \Big| dx$$

$$\begin{aligned} &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)|\,|g(y)| dy \right) dx \\ &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |f(x-y)| dx \right) |g(y)|\, dy = ||f||_1 ||g||_1. \end{aligned}$$

3) is an easy consequence of Fubini's theorem and 6):

$$\begin{aligned} \widehat{f * g}(k) &= \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} f(x-y)\, g(y)\, dy \right) e^{-ik\cdot x} dx \\ &= \int_{\mathbb{R}^n} \underbrace{\left(\int_{\mathbb{R}^n} f(x-y) e^{-ik\cdot x} dx \right)}_{= e^{-ik\cdot y}\widehat{f}(k) \text{ by 6)}} g(y)\, dy = \widehat{f}(k)\, \widehat{g}(k). \end{aligned}$$

To establish 4), we start from the expression on the left hand side, substitute the reconstruction formula for $g(x)$, and apply Fubini's theorem:

$$\begin{aligned} \widehat{fg}(k) &= \int_{\mathbb{R}^n} f(x) \underbrace{g(x)}_{=(2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{g}(\ell) e^{i\ell\cdot x} d\ell} e^{-ik\cdot x} dx \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} \underbrace{\int_{\mathbb{R}^n} f(x) e^{-i(k-\ell)\cdot x} dx}_{=\widehat{f}(k-\ell)} \widehat{g}(\ell)\, d\ell \\ &= (2\pi)^{-n} (\widehat{f} * \widehat{g})(k). \end{aligned}$$

To show 5), apply the reconstruction formula at the point $-x$ to infer

$$f(-x) = (2\pi)^{-n} \underbrace{\int_{\mathbb{R}^n} \widehat{f}(k) e^{i(-x)\cdot k} dk}_{=\widehat{\widehat{f}}(x)}.$$

7) is easy to show under the additional assumption that f vanishes outside some bounded set: by an integration by parts with respect to x_j, together with the fact that by the additional

assumption no boundary terms appear,

$$\begin{aligned}
\widehat{\frac{\partial f}{\partial x_j}}(k) &= \int_{\mathbb{R}^n} e^{-ik\cdot x}\frac{\partial f}{\partial x_j}(x)\,dx \\
&= \int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}} e^{-ik\cdot x}\frac{\partial f}{\partial x_j}(x)\,dx_j\right)\prod_{\ell\neq j} dx_\ell \\
&= \int_{\mathbb{R}^{n-1}}\left(-\int_{\mathbb{R}}(-ik_j e^{-ik\cdot x})f(x)\,dx_j\right)\prod_{\ell\neq j} dx_\ell \\
&= ik_j\widehat{f}(k).
\end{aligned}$$

To establish 7) under the natural assumptions given in the theorem is less obvious.[23] We recommend to omit the argument, given in Appendix A, at first reading. The proof of properties 1)-7) is complete.

[23] One could appeal to the fact that the functions satisfying the additional assumption are dense in the Sobolev space $W^{1,1}(\mathbb{R}^n)$, but we do not wish to assume familiarity with Sobolev spaces here.