Exercise 10

Weyl's criterion: if the sequence $(\xi_n)_n \in [0,1)$ is equidistributed, then $\forall\, k \in \mathbb{Z} \setminus \{0\}$

$$\frac{1}{N} \sum_{n=1}^{N} e^{2\pi i k \xi_n} \longrightarrow 0 \quad \text{as } N \to +\infty$$

Exercise 11 (Gibbs phenomenon)

Let f be the sawtooth function defined by $f(x) = (\pi - x)/2$ on $(0, 2\pi)$ with $f(0) = 0$ and extended by periodicity to $\mathbb{R}$. The Fourier series of f is

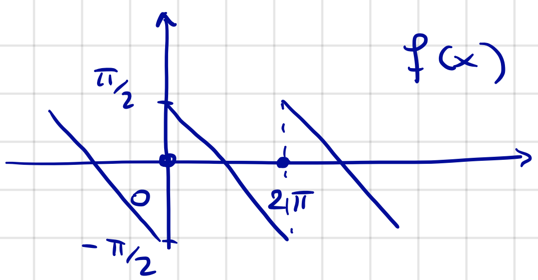
$$f(x) = \sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

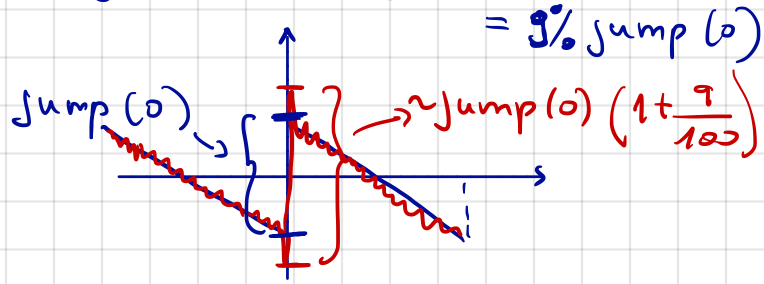
f has a jump discontinuity at 0:

$$\lim_{x \to 0^+} f(x) - \lim_{x \to 0^-} f(x) = \pi/2 - (-\pi/2) = \pi$$

Show that $\max\limits_{0 \le x \le \pi/N} S_N(f)(x) - \pi/2 = \displaystyle\int_0^{\pi} \frac{\sin t}{t}\, dt - \frac{\pi}{2} + O\!\left(\frac{1}{N}\right)$.

Note that numerically one can see that $\int_0^{\pi} \sin t / t \, dt - \pi/2 \simeq 9\%\, \pi =$
$= 9\%$ jump (0)





<u>Hint</u>: use that $\displaystyle\sum_{n=1}^{N} \frac{\sin nx}{n} = \frac{1}{2} \int_0^x \left(D_N(t) - 1\right) dt$.